

ELECTRICAL ENGINEERING

Electromagnetic Theory



Comprehensive Theory
with Solved Examples and Practice Questions





MADE EASY Publications Pvt. Ltd.

Corporate Office: 44-A/4, Kalu Sarai (Near Hauz Khas Metro Station), New Delhi-110016 | **Ph. :** 9021300500

Email : infomep@madeeasy.in | **Web :** www.madeeasypublications.org

Electromagnetic Theory

Copyright © by MADE EASY Publications Pvt. Ltd.
All rights are reserved. No part of this publication may be reproduced, stored in or introduced into a retrieval system, or transmitted in any form or by any means (electronic, mechanical, photo-copying, recording or otherwise), without the prior written permission of the above mentioned publisher of this book.



MADE EASY Publications Pvt. Ltd. has taken due care in collecting the data and providing the solutions, before publishing this book. In spite of this, if any inaccuracy or printing error occurs then **MADE EASY Publications Pvt. Ltd.** owes no responsibility. We will be grateful if you could point out any such error. Your suggestions will be appreciated.

EDITIONS

First Edition : 2015
Second Edition : 2016
Third Edition : 2017
Fourth Edition : 2018
Fifth Edition : 2019
Sixth Edition : 2020
Seventh Edition : 2021
Eighth Edition : 2022
Ninth Edition : 2023
Tenth Edition : 2024

Eleventh Edition : 2025

CONTENTS

Electromagnetic Theory

CHAPTER 1

Vector Analysis 1-29

- 1.1 Introduction 1
- 1.2 Coordinate Systems 6
- 1.3 Vector Calculus 12
 - Objective Brain Teasers* 26
 - Conventional Brain Teasers* 28

CHAPTER 2

Electrostatics 30-75

- 2.1 Introduction 30
- 2.2 Gauss's Law - Maxwell's Equation 30
- 2.3 Electric Flux Density 32
- 2.4 Applications of Gauss's Law 33
- 2.5 Electric Field Intensity 38
- 2.6 Coulomb's Law 39
- 2.7 Electric fields due to Charge Distributions 41
- 2.8 Electric Potential 44
- 2.9 Electric Field as the Gradient of the Potential-Maxwell's Equation 47
- 2.10 Potential due to Electric Dipole 49
- 2.11 Energy Density in Electrostatic Field 51
- 2.12 Current and Current Density 53
- 2.13 Continuity Equation 55
- 2.14 Boundary Conditions 57
- 2.15 Poisson's and Laplace's Equations 62
- 2.16 Capacitance 64
- 2.17 Induced Charge and Method of Images 67

- 2.18 Permittivity 68
 - Objective Brain Teasers* 69
 - Conventional Brain Teasers* 72

CHAPTER 3

Magnetostatics 76-99

- 3.1 Introduction 76
- 3.2 Biot-Savart's Law 77
- 3.3 Ampere's Circuit Law-Maxwell's Equation 82
- 3.4 Magnetic Flux Density - Maxwell's Equation 84
- 3.5 Magnetic Scalar and Vector Potentials 86
- 3.6 Forces Due To Magnetic Fields 87
- 3.7 Magnetic Boundary Conditions 92
- 3.8 Permeability 94
 - Objective Brain Teasers* 96
 - Conventional Brain Teasers* 98

CHAPTER 4

Time-Varying Electromagnetic Fields 100-109

- 4.1 Introduction 100
- 4.2 Maxwell's Equations for Static EM Fields 100
- 4.3 Faraday's Law of Induction 101
- 4.4 Transformer and Motional EMFs 102
- 4.5 Displacement Current 105
- 4.6 Maxwell's Equations In Final Forms 106
- 4.7 Time-Varying Potentials 106
 - Objective Brain Teasers* 107
 - Conventional Brain Teasers* 108

■■■■

Vector Analysis

1.1 INTRODUCTION

Vector analysis is a concise language or mathematical shorthand which greatly facilitates the analysis of electric and magnetic fields. The quantities of interest appearing in the study of EM theory can almost be classified as either a scalar or a vector.

Quantities that can be described by a magnitude alone are called **scalars**. Distance, temperature, mass etc. are examples of scalar quantities. Other quantities, called **vectors**, require both a magnitude and a direction to fully characterize them. Examples of vector quantities include velocity, force, acceleration etc.

In electromagnetics, we frequently use the concept of a **field**. A field is a function that assigns a particular physical quantity to every point in a region. In general, a field varies with both position and time. There are scalar fields and vector fields. Temperature distribution in a room and electric potential are examples of scalar fields. Electric field and magnetic flux density are examples of vector fields.

Note: Vectors are denoted by an arrow over a letter (\vec{A}) and scalars are denoted by simple letter (A).

1.1.1 Unit Vector

A unit vector \hat{a}_A along \vec{A} is defined as a vector whose magnitude is unity (*i.e.*, 1) and its direction is along \vec{A} , that is

$$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A} \quad \dots(1.1)$$

Thus we can write \vec{A} as $\vec{A} = A\hat{a}_A = |\vec{A}|\hat{a}_A \quad \dots(1.2)$

Remember: Any vector can be written as product of its magnitude and its unit vector.

1.1.2 Vector Addition and Subtraction

Two vectors \vec{A} and \vec{B} can be added together to give another vector \vec{C} ; that is,

$$\vec{C} = \vec{A} + \vec{B} \quad \dots(1.3)$$

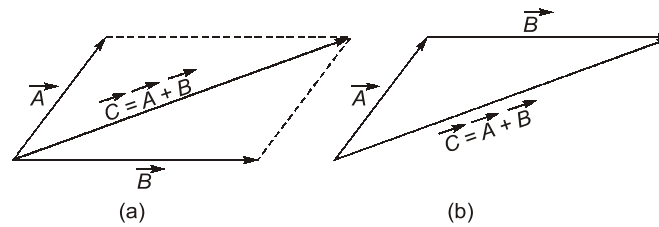


Fig.: Vector addition (a) parallelogram rule, (b) head-to-tail rule.



- $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ (Commutative law).
- $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$ (Associative law)

Vector subtraction is similarly carried out as

$$\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \quad \dots(1.4)$$



Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule.



- $k(\vec{A} + \vec{B}) = k\vec{A} + k\vec{B}$ (Distributive law)
- $\frac{\vec{A} + \vec{B}}{k} = \frac{1}{k}\vec{A} + \frac{1}{k}\vec{B}$

1.1.3 Position and Distance Vectors:

A point P in Cartesian coordinates may be represented by (x, y, z) .

The position vector \vec{r}_p (or radius vector) of point P is defined as the directed distance from origin O to P .



Fig.: Illustration of position vector $\vec{r}_p = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$

The distance vector is the displacement from one point to another.

Consider point P with position vector \vec{r}_p and point Q with position vector \vec{r}_q . The displacement from P to Q is written as

$$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p \quad \dots(1.6)$$

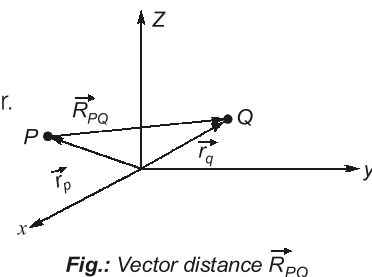


Fig.: Vector distance \vec{R}_{PQ}

EXAMPLE : 1.1

Point P and Q are located at $(0, 2, 4)$ and $(-3, 1, 5)$. Calculate

- The position vector P
- The distance vector from P to Q
- The distance between P and Q
- A vector parallel to PQ with magnitude of 10.

Solution:

$$\begin{aligned} \text{(a)} \quad \vec{r}_P &= 0\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z = 2\hat{a}_y + 4\hat{a}_z \\ \text{(b)} \quad \vec{R}_{PQ} &= \vec{r}_Q - \vec{r}_P = (-3, 1, 5) - (0, 2, 4) = (-3, -1, 1) \\ &= -3\hat{a}_x - \hat{a}_y + \hat{a}_z \end{aligned}$$

(c) The distance between P and Q is the magnitude of \vec{R}_{PQ} ; that is

$$d = |\vec{R}_{PQ}| = \sqrt{9+1+1} = 3.317$$

(d) Let the required vector be \vec{A} , then

$$\vec{A} = A\hat{a}_A$$

where $A = 10$ is magnitude of \vec{A}

$$\text{and} \quad \hat{a}_A = \frac{\vec{R}_{PQ}}{|\vec{R}_{PQ}|} = \pm \frac{(-3, -1, 1)}{3.317}$$

$$\text{then} \quad \vec{A} = \pm \frac{10(-3, -1, 1)}{3.317} = \pm (-9.045\hat{a}_x - 3.015\hat{a}_y + 3.015\hat{a}_z)$$

1.1.4 Vector Multiplication

When two vectors are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication.

- Scalar (or dot) product : $\vec{A} \cdot \vec{B}$
- Vector (or cross) product : $\vec{A} \times \vec{B}$

Multiplication of three vectors \vec{A} , \vec{B} , \vec{C} can result in either

- Scalar triple product : $\vec{A} \cdot (\vec{B} \times \vec{C})$
- Vector triple product : $\vec{A} \times (\vec{B} \times \vec{C})$

1. Dot Product

The dot product, or the scalar product of two vectors \vec{A} and \vec{B} , written as $\vec{A} \cdot \vec{B}$ is defined geometrically as the product of the magnitudes of \vec{A} and \vec{B} and the cosine of the angle between them.

$$\text{Thus} \quad \vec{A} \cdot \vec{B} = AB \cos \theta_{AB} \quad \dots(1.7)$$

Where θ_{AB} is the smaller angle between \vec{A} and \vec{B} . **The result of $\vec{A} \cdot \vec{B}$ is called either the scalar product because it is scalar, or the dot product due to the dot sign.**

$$\text{If} \quad \vec{A} = (A_x, A_y, A_z) \quad \text{and} \quad \vec{B} = (B_x, B_y, B_z)$$

$$\text{then} \quad \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad \dots(1.8)$$

The dot product is commutative and distributive.

Note: Two vectors \vec{A} and \vec{B} are said to be orthogonal (or perpendicular) with each other if $\vec{A} \cdot \vec{B} = 0$

The dot product obeys the following:

Law

Expression

Cumulative $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$... (1.9)

Distributive $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$... (1.10)

$$\vec{A} \cdot \vec{A} = |\vec{A}|^2 = A^2 \quad \dots (1.11)$$

Also note that:

$$\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0 \quad \dots (1.12)$$

$$\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1 \quad \dots (1.13)$$

2. Cross Product:

The cross product of two vectors \vec{A} and \vec{B} , written as $\vec{A} \times \vec{B}$, is a vector quantity whose magnitude is the area of the parallelepiped formed by \vec{A} and \vec{B} and is in the direction of advance of the right-handed screw as \vec{A} is turned into \vec{B} .

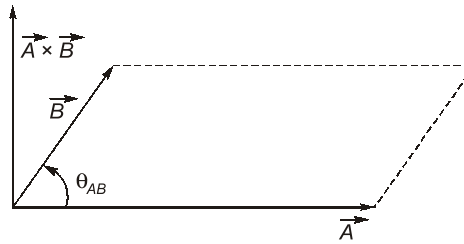


Fig.: The cross product of \vec{A} and \vec{B} is a vector with magnitude equal to the area of parallelogram and the direction as indicated

Thus $\vec{A} \times \vec{B} = AB \sin \theta_{AB} \hat{a}_n$... (1.14)

where \hat{a}_n is a unit vector normal to the plane containing \vec{A} and \vec{B} .

The vector multiplication of equation (1.14) is called **cross product** due to the cross sign. It is also called **vector product** because the result is a vector.

If $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$ then :

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \dots (1.15)$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{a}_x + (A_z B_x - A_x B_z) \hat{a}_y + (A_x B_y - A_y B_x) \hat{a}_z \quad \dots (1.16)$$

Which is obtained by 'crossing' terms in cyclic permutation, hence the name cross product.

Vector product is not commutative and not associative but vector product is distributive.

Note that the cross product has the following properties

1. It is not commutative:

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A} \quad \dots (1.17)$$

Note: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

2. It is not associative:

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C} \quad \dots(1.18)$$

3. It is distributive:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad \dots(1.19)$$

Note: $\vec{A} \times \vec{A} = 0$

4. Also note that

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z \quad \dots(1.20)$$

$$\hat{a}_y \times \hat{a}_z = \hat{a}_x \quad \dots(1.21)$$

$$\hat{a}_z \times \hat{a}_x = \hat{a}_y \quad \dots(1.22)$$



If $\vec{A} \times \vec{B} = 0$, then $\sin \theta_{AB} = 0^\circ$ or 180° ; this shows that \vec{A} and \vec{B} are parallel or antiparallel to each other

3. Scalar Triple Product:

Given three vectors \vec{A} , \vec{B} , and \vec{C} , we define scalar triple product as,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \quad \dots(1.23)$$

If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ and $\vec{C} = (C_x, C_y, C_z)$, then $\vec{A} \cdot (\vec{B} \times \vec{C})$ is the volume of a parallelepiped having \vec{A} , \vec{B} , and \vec{C} as edges and is easily obtained by finding the determinant of the 3×3 matrix formed by \vec{A} , \vec{B} , and \vec{C} ; that is

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad \dots(1.24)$$

Since the result of this vector multiplication is scalar these two equations are called the scalar triple product.

4. Vector Triple Product:

For vectors \vec{A} , \vec{B} , and \vec{C} , we define the vector triple product as

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \quad \dots(1.25)$$

This is obtained using the “bac-cab” rule.

EXAMPLE : 1.2

Three field quantities are given by $\vec{P} = 2\hat{a}_x - \hat{a}_z$ and $\vec{Q} = 2\hat{a}_x - \hat{a}_y + 2\hat{a}_z$, $\vec{R} = 2\hat{a}_x - 3\hat{a}_y + \hat{a}_z$. Determine:

- (a) $(\vec{P} + \vec{Q}) \times (\vec{P} - \vec{Q})$ (b) $\vec{Q} \cdot (\vec{R} \times \vec{P})$ (c) $\vec{P} \cdot (\vec{Q} \times \vec{R})$
(d) $\sin \theta_{QR}$ (e) $\vec{P} \times (\vec{Q} \times \vec{R})$

(f) A unit vector perpendicular to both \vec{Q} and \vec{R}

(g) The component of \vec{P} along \vec{Q}

Solution:

$$(a) \quad (\vec{P} + \vec{Q}) \times (\vec{P} - \vec{Q}) = 2(\vec{Q} \times \vec{P})$$

$$= 2 \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = 2(1-0)\hat{a}_x + 2(4+2)\hat{a}_y + 2(0+2)\hat{a}_z = 2\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z$$

$$(b) \quad \vec{Q} \cdot (\vec{R} \times \vec{P}) = (2, -1, 2) \cdot \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} \\ = (2, -1, 2) \cdot (3, 4, 6) = 6 - 4 + 12 = 14$$

$$\text{Alternatively:} \quad \vec{Q} \cdot (\vec{R} \times \vec{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 14$$

$$(c) \quad \vec{P} \cdot (\vec{Q} \times \vec{R}) = \vec{Q} \cdot (\vec{R} \times \vec{P}) = 14$$

$$(d) \quad \sin \theta_{QR} = \frac{|\vec{Q} \times \vec{R}|}{|\vec{Q}| |\vec{R}|} = \frac{\sqrt{45}}{3\sqrt{14}} = 0.5976$$

$$(e) \quad \vec{P} \times (\vec{Q} \times \vec{R}) = \vec{Q}(\vec{P} \cdot \vec{R}) - \vec{R}(\vec{P} \cdot \vec{Q}) \\ = (2, -1, 2)(4 + 0 - 1) - (2, -3, 1)(4 + 0 - 2) = (2, 3, 4) = 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$$

(f) A unit vector perpendicular to both \vec{Q} and \vec{R} is given by

$$\hat{a}_n = \pm \frac{\vec{Q} \times \vec{R}}{|\vec{Q} \times \vec{R}|} = \frac{\pm(5, 2, -4)}{\sqrt{45}} = \pm(0.745, 0.298, -0.596) \\ \hat{a}_n = \pm(0.745\hat{a}_x + 0.298\hat{a}_y - 0.596\hat{a}_z)$$

$$\text{Note that,} \quad |\hat{a}_n| = 1, \quad \hat{a}_n \cdot \vec{Q} = \hat{a}_n \cdot \vec{R} = 0$$

The component of \vec{P} along \vec{Q} is

$$\vec{P}_Q = |\vec{P}| \cos \theta_{PQ} \hat{a}_Q = (\vec{P} \cdot \hat{a}_Q) \hat{a}_Q = \frac{(\vec{P} \cdot \vec{Q}) \vec{Q}}{|\vec{Q}|^2} = \frac{(4+0-2)(2, -1, 2)}{(4+1+4)} \\ = \frac{2}{9}(2, -1, 2) = 0.4444\hat{a}_x - 0.2222\hat{a}_y + 0.4444\hat{a}_z$$

1.2 COORDINATE SYSTEMS

A coordinate system defines points of reference from which specific vector directions may be defined.

Depending on the geometry of the application, one coordinate system may lead to more efficient vector definitions than others. The three most commonly used co-ordinate systems used in the study of electromagnetics are rectangular coordinates (or Cartesian coordinates), cylindrical coordinates and spherical coordinates.

Note: An orthogonal system is one in which the coordinates are mutually perpendicular



OBJECTIVE BRAIN TEASERS

- Q.1** A point is represented in Cartesian coordinates as $P(3, 4, 5)$, the radial component ρ in cylindrical coordinates will be
 (a) less than (b) greater than
 (c) equal to (d) unrelated to r in spherical coordinates.
- Q.2** Consider a closed surface S surrounding a volume V . If \vec{r} is the position vector of a point inside S , with \hat{n} the unit normal on S , the value of the integral $\oint_S 2\vec{r} \cdot \hat{n} dS$ is
 (a) 3 V (b) 2 V
 (c) 6 V (d) 4 V
- Q.3** Consider a vector $\vec{E} = z\hat{a}_x + (x + y)\hat{a}_y$, the z component of the vector in cylindrical coordinates will be
 (a) z
 (b) $z \cos \phi + (x + y) \sin \phi$
 (c) $-z \sin \phi + (x + y) \cos \phi$
 (d) zero
- Q.4** The direction of vector \vec{A} is radially outward from the origin, with $|\vec{A}| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla \cdot \vec{A} = 0$ is
 (a) -2 (b) 2
 (c) 1 (d) 0
- Q.5** Let a point in spherical and cylindrical coordinates are (r, θ, ϕ) and (ρ, ϕ, z) . The radial component r in spherical coordinates is related to components in cylindrical coordinates as
 (a) ρ (b) $\rho \cos \phi$
 (c) $z \tan^{-1} \phi$ (d) $(\rho^2 + z^2)^{1/2}$
- Q.6** Given the vector

$$\vec{A} = (\cos x)(\sin y)\hat{a}_x + (\sin x)(\cos y)\hat{a}_y,$$
 where \hat{a}_x, \hat{a}_y denote unit vectors along x, y directions, respectively. The magnitude of curl of \vec{A} is
 (a) 0 (b) 1
 (c) -1 (d) 2
- Q.7** Given a vector field $\vec{A} = 2r \cos \phi \hat{a}_r$ in cylindrical coordinates. For the contour as shown below, $\oint \vec{A} \cdot d\vec{l}$ is
 (a) 1
 (b) $1 - (\pi/2)$
 (c) $1 + (\pi/2)$
 (d) -1
-
- Q.8**
-
- Circle of unit radius
- What is the value of the integral $\int_c d\vec{l}$ along the curve c (c is the curve ABCD in the direction of the arrow)?
 (a) $2(\hat{a}_x + \hat{a}_y)/\sqrt{2}$
 (b) $-2(\hat{a}_x + \hat{a}_y)/\sqrt{2}$
 (c) $2\hat{a}_x$
 (d) $-2\hat{a}_y$

ANSWERS KEY

1. (a) 2. (c) 3. (d) 4. (a) 5. (d)
 6. (a) 7. (a) 8. (d)

HINTS & EXPLANATIONS

1. (a)

$$P(3, 4, 5), \quad \rho = \sqrt{3^2 + 4^2} = 5$$

$$r = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$$

$$\text{So, } r > \rho$$

2. (c)

Position vector,

$$\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

$$\oint_S 2\vec{r} \cdot \vec{n} dS = 2 \iiint_V (\vec{\nabla} \cdot \vec{r}) dV$$

$$\vec{\nabla} \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\oint_S 2\vec{r} \cdot \vec{n} dS = 6 \iiint_V dV = 6V$$

3. (d)

$$\vec{E} = z\hat{a}_x + (x+y)\hat{a}_y$$

Z component of Cartesian and cylindrical component are equal.

Hence, z component is 0.

4. (a)

$$r^2 = x^2 + y^2 + z^2$$

$$|\vec{A}| = kr^n$$

$$\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

$$\nabla \cdot \vec{A} = \nabla \cdot (kr^n \hat{a}_r) = k \nabla \cdot (r^n \hat{a}_r)$$

$$\nabla \cdot \vec{A} = k \nabla \cdot \left(r^n \frac{\hat{a}_r}{r_1} \right)$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^n) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^{n+2})$$

$$\text{for } \nabla \cdot \vec{A} = 0, r^{n+2} = \text{constant}$$

$$\Rightarrow n + 2 = 0$$

5. (d)

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\rho = \sqrt{x^2 + y^2} \Rightarrow r^2 = x^2 + y^2$$

$$\Rightarrow r = \sqrt{\rho^2 + z^2}$$

6. (a)

$$\vec{A} = \cos x \sin y \hat{a}_x + \sin x \cos y \hat{a}_y$$

$$\vec{\nabla} \times \vec{A} = \begin{bmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x \sin y & \sin x \cos y & 0 \end{bmatrix}$$

$$\vec{\nabla} \times \vec{A} = 0 \cdot \hat{a}_x + (-0 \cdot \hat{a}_y) + [\cos x \cos y - \cos x \cos y] \hat{a}_z$$

$$\Rightarrow \vec{\nabla} \times \vec{A} = 0 \Rightarrow |\vec{\nabla} \times \vec{A}| = 0$$

7. (a)

$$\vec{A} = 2r \cos \phi \hat{a}_r$$

for Stokes's theorem

$$\oint_S \vec{A} \cdot d\vec{l} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r} \begin{bmatrix} \hat{a}_r & r\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 2r \cos \phi & 0 & 0 \end{bmatrix} = +2 \sin \phi \hat{a}_z$$

$$\iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = \int_0^{\pi/2} \int_0^1 +2r \sin \phi dr d\phi = \int_0^{\pi/2} \left[\frac{2r^2}{2} \right]_0^1 \sin \phi d\phi$$

$$\Rightarrow \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = [-\cos \phi]_0^{\pi/2} = 1$$

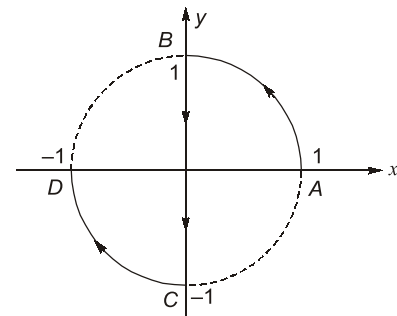
$$\Rightarrow \oint_S \vec{A} \cdot d\vec{l} = 1$$

8. (d)

$$\int_C d\vec{l} = \int_{AB} d\vec{l} + \int_{BC} d\vec{l} + \int_{CD} d\vec{l}$$

$$\text{As } \int_{AB} d\vec{l} \text{ and } \int_{CD} d\vec{l}$$

are in opposite direction



$$\text{i.e., } \int_{AB} d\vec{l} = - \int_{CD} d\vec{l}$$

$$\Rightarrow \int_C d\vec{l} = \int_{BC} d\vec{l} = -2\hat{a}_y$$



CONVENTIONAL BRAIN TEASERS

Q.1 Find the area of the curved surface of a right circular cylinder of radius r and height h using cylindrical coordinates.

1. (Sol.)

radius of cylinder, ' r '; height of cylinder, ' h '

Curved surface area,
$$A = \int_0^{2\pi} \int_0^h r \, dz \, d\phi = r \int_0^{2\pi} h \, d\phi = 2\pi rh$$

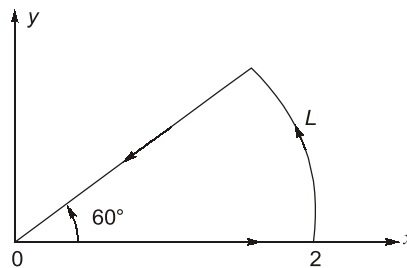
Q.2 Calculate the volume in spherical coordinates defined by

$$1 \leq r \leq 2 \text{ m}, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}$$

2. (Sol.)

$$\begin{aligned} \text{Volume, } V &= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 r^2 \sin\theta \, dr \, d\theta \, d\phi = \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{r^3}{3} \right]_1^2 \sin\theta \, d\theta \, d\phi \\ V &= \frac{7}{3} \int_0^{\pi/2} [-\cos\theta]_0^{\pi/2} \, d\phi = \frac{7}{3} [\phi]_0^{\pi/2} = \frac{7}{6} \pi \text{ m}^3 \end{aligned}$$

Q.3 Calculate the circulation of $A = \rho \cos\phi \hat{a}_\rho + z \sin\phi \hat{a}_z$ around the edge L of the wedge defined by $0 < \rho < 2$, $0 \leq \phi \leq 60^\circ$, $z = 0$ and shown in figure.



3. (Sol.)

$$A = \rho \cos\phi \hat{a}_\rho + z \sin\phi \hat{a}_z$$

$$\begin{aligned} \vec{\nabla} \times \vec{A} &= \frac{1}{\rho} \begin{bmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho \cos\phi & 0 & z \sin\phi \end{bmatrix} = \frac{1}{\rho} [(z \cos\phi) \hat{a}_\rho - \rho \hat{a}_\phi [0] + (\rho \sin\phi) \hat{a}_z] \\ &= \frac{z \cos\phi}{\rho} \hat{a}_\rho + \frac{\rho \sin\phi}{\rho} \hat{a}_z \end{aligned}$$