POSTAL BOOK PACKAGE 2024

MADE EASY PUBLICATIONS

ELECTRONICS ENGINEERING

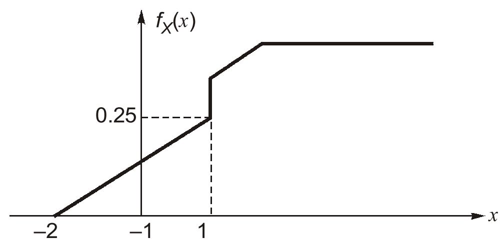
CONVENTIONAL Practice Sets

CONTENTS

COMMUNICATION SYSTEMS

1 CHAPTER Theory of Random Variable and Noise

Q.1 Define PDF and summarise its important properties. Also calculate the probability of outcome of a Random Variable (RV) X having $X \leq 1$ for the following PDF curve of RV as shown.



Solution:

Probability density function specifies the probability of a random variable taking a particular value.

The Probability Density Function (PDF) which is generally denoted by $f_X(x)$ or $P_X(x)$ or $\rho_X(x)$ is defined in terms of the Cumulative Distribution Function (CDF) $F_X(x)$ as,

$$\boxed{\text{PDF} = f_X(x) = \frac{d}{dx} F_X(x)} \quad \text{...(i)}$$

The PDF has the following properties:

(i) $f_X(x) \geq 0$ for all x

This results from the fact that probability cannot be negative. Also, $F_X(x)$ increases monotonically, as x increases, more outcomes are included in the prob. of occurrence represented by $F_X(x)$.

(ii) Area under the PDF curve is always equal to unity.

i.e. $$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

(iii) The CDF is obtained by the result

$$\text{CDF} = \int_{-\infty}^{x} f_X(x) dx$$

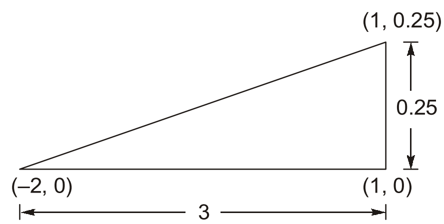
(iv) Probability of occurrence of the value of random variable between the limits of x_1 and x_2 is given by,

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x) dx$$

Now consider the given PDF curve, since we have to find $P(x \leq 1)$ so,

Equation for the PDF curve for $x \leq 1$ is,

$$f_X(x) = \left(\frac{1}{12}x + \frac{1}{6}\right)$$



Now, $P(x \leq 1)$

$$= P(-2 < x < 1) = \int_{-2}^{1}\left(\frac{1}{12}x + \frac{1}{6}\right)dx = \left[\frac{1}{12}\cdot\frac{x^2}{2} + \frac{1}{6}x\right]_{-2}^{1} = \frac{3}{8}$$

$$\therefore \qquad P(x \leq 1) = \frac{3}{8}$$

Q.2 Find the cumulative distribution function $F(x)$ corresponding to the PDF $f(x) = \dfrac{1}{\pi(1+x^2)}, -\infty < x < \infty$.

Solution:

Given

$$f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$$

$$F(x) = P(X \leq x)$$

$$= \int_{-\infty}^{x} f(x)dx = \frac{1}{\pi}\int_{-\infty}^{x}\frac{dx}{1+x^2} = \frac{1}{\pi}\left[\tan^{-1}x\right]_{-\infty}^{x} = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}x\right)$$

Q.3 Given the random variable X with density function

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the PDF of $Y = 8X^3$.

Solution:

$y = 8x^3$ is an increasing function in (0, 1)

Given $\quad y = 8x^3$

$\Rightarrow \quad x^3 = \dfrac{y}{8}$

$\Rightarrow \quad x = \left(\dfrac{y}{8}\right)^{1/3} = \dfrac{1}{2}y^{1/3}$

and $\quad f_X(x) = 2x, \qquad 0 < x < 1$

$$f_X(y) = \frac{2y^{1/3}}{2} = \frac{y^{1/3}}{3}$$

$f_Y(y) =$ ...(i)

Given $\quad x = \left(\dfrac{y}{8}\right)^{1/3} = \dfrac{1}{2}y^{1/3} \quad \Rightarrow \quad \dfrac{dx}{dy} = \dfrac{1}{6}y^{-2/3}$

Using it in (i)

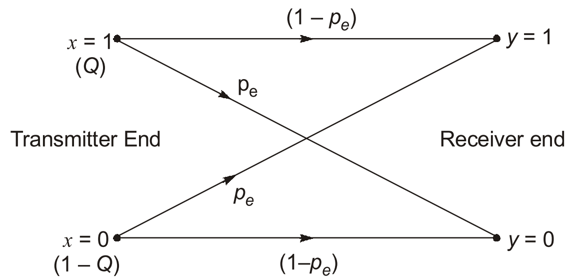
$$f_Y(y) = y^{1/3}\frac{1}{6}y^{-2/3} = \frac{1}{6}y^{-1/3} = \frac{1}{6}\frac{1}{y^{1/3}} = \frac{1}{6\sqrt[3]{y}}$$

The range for x is $0 < x < 1$

When $x = 0,\ y = 8 \times 0 = 0$ and $x = 1,\ y = 8 \times 1^3 = 8$

$$f_Y(y) = \frac{1}{6\sqrt[3]{y}}, \qquad 0 < y < 8$$

Q.4 A BSC (Binary Symmetric Channel) error probability is P_e. The probability of transmitting '1' is Q, and that of transmitting '0' is $(1 - Q)$ as in figure below. Calculate the probabilities of receiving 1 and 0 at the receiver?



Solution:

If x and y are the transmitted digit and the received digit respectively, then for a BSC,

$$P_{y|x}(0|1) = P_{y|x}(1|0) = P_e$$
$$P_{y|x}(0|0) = P_{y|x}(1|1) = 1 - P_e$$

Also, $P_x(1) = Q$ and $P_x(0) = 1 - Q$

We have to find, $P_y(1)$ and $P_y(0) = ?$

$\therefore$ $$P_y(1) = P_x(0)\,P_{y|x}(1|0) + P_x(1)\,P_{y|x}(1|1) = (1 - Q)P_e + Q(1 - P_e)$$

also, $$P_y(0) = P_x(0)P_{y|x}(0|0) + P_x(1)\,P_{y|x}(0|1) = (1 - Q)(1 - P_e) + QP_e$$

Q.5 For the triangular distribution

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance.

Solution:

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot x\,dx + \int_1^2 x(2 - x)\,dx = \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 x\,dx + \int_1^2 x^2(2 - x)\,dx$$

$$= \int_0^1 x^3\,dx + \int_1^2 (2x^2 - x^3)\,dx = \left[\frac{x^4}{4}\right]_0^1 + \left[2\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right] = \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{7}{6}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{7}{6} - (1)^2 = \frac{1}{6}$$

Q.6 The joint density function of two continuous random variables is given by

$$f(x,\ y) = \begin{cases} xy/8, & 0 < x < 2,\ 1 < y < 3 \\ 0, & \text{otherwise} \end{cases}$$

Find (a) $E(X)$, (b) $E(Y)$ and (c) $E(2X + 2Y)$.

Solution:

(a) $$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dx\,dy = \int_{x=0}^{2}\int_{y=1}^{3} x(xy/8)\,dx\,dy = \frac{4}{3}$$

(b) $$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy = \int_{x=0}^{2}\int_{y=1}^{3} y(xy/8)\,dx\,dy = \frac{13}{6}$$

(c) $$E(2X+3Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (2x+3y)\,dx\,dy = \int_{x=0}^{2}\int_{y=1}^{3} (2x+3y)(xy/8)dx\,dy = \frac{55}{6}$$

Q.7 Let z be a random variable with probability density function $f_Z(z) = \frac{1}{2}$ in the range $-1 \le z \le 1$. Let the random variable $x = z$ and the random variable $y = z^2$. Obviously x and y are not independent since $x^2 = y$. Show that x and y are uncorrelated.

Solution:

We have, $$E(z) = \int_{-1}^{1} z\cdot f_Z(z)\,dz$$

$\Rightarrow$ $$E(z) = \frac{1}{4}\left[z^2\right]_{-1}^{1} = 0$$

Since, $x = z$, so $E(x) = E(z) = 0$

Since, $y = z^2$ so $E(y) = E(z^2)$

So that, $$E(y) = \int_{-1}^{1}\frac{1}{2}z^2 dz = \frac{1}{6}\left[z^3\right]_{-1}^{1} = \frac{1}{3}$$

We know that, the co-variance 'μ' of two RVs x and y is defined as,

$$\mu = E\{(x-m_x)(y-m_y)\}$$

$$= E\left\{(x)\left(y-\frac{1}{3}\right)\right\} = E\left\{xy - \frac{1}{3}x\right\} = E\left\{z^3 - \frac{z}{3}\right\} = \int_{-1}^{1}\frac{1}{2}\left(z^3 - \frac{z}{3}\right)dz$$

$$\mu = 0$$

Now, correlation coefficient between the variables x and y is defined by quantity 'ρ' as,

$$\rho = \frac{\mu}{\sigma_x\sigma_y} = 0$$

So, we can say that these RV's X and Y are uncorrelated.

Q.8 A WSS random process $x(t)$ is applied to the input of an LTI system with impulse response

$$h(t) = 3e^{-2t}\,u(t)$$

Find the mean value of the output $y(t)$ of the system, if $E[x(t)] = 2$. Here E [·] denotes the expectation operator.

Solution:

The output $y(t)$ is the convolution of the input $x(t)$ and the impulse response $h(t)$.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\cdot d\tau$$

$$\therefore \quad E[y(t)] = \int_{-\infty}^{\infty} h(\tau)\cdot E[x(t-\tau)]\cdot d\tau$$

$$E[y(t)] = H(0)\times E[x(t)]$$

$$\boxed{E[y(t)] = E[x(t)]\cdot H(0)}$$

where, $H(0) = H(\omega)\big|_{\omega=0}$ and $H(\omega)$ = Fourier transform of $h(t)$

Given $E[x(t)] = 2$, $\quad h(t) = 3e^{-2t}u(t)$

Taking Fourier transform, $\quad H(\omega) = \dfrac{3}{2+j\omega} \Rightarrow H(0) = \dfrac{3}{2}$

$$E[y(t)] = 2\times\frac{3}{2} = 3$$

Q.9 Suppose that two signals $s_1(t)$ and $s_2(t)$ are orthogonal over the interval $(0, T)$. A sample function $n(t)$ of a zero-mean white noise process is correlated with $s_1(t)$ and $s_2(t)$ separately, to yield the following variables:

$$n_1 = \int_0^T s_1(t)\,n(t)\,dt \quad \text{and} \quad n_2 = \int_0^T s_2(t)\,n(t)\,dt$$

Prove that n_1 and n_2 are orthogonal.

Solution:

$$E[n_1 n_2] = E\left[\int_0^T s_1(u)n(u)du \int_0^T s_2(v)n(v)dv\right]$$

$$= \int_0^T\int_0^T s_1(u)s_2(v)E[n(u)n(v)]\,du\,dv$$

$n(t)$ is a white noise process.

So, $\quad R_N(\tau) = \dfrac{N_0}{2}\delta(\tau)$

$$E[n(u)\,n(v)] = \frac{N_0}{2}\delta(u-v)$$

Hence, $\quad E[n_1 n_2] = \dfrac{N_0}{2}\displaystyle\int_0^T\int_0^T s_1(u)s_2(v)\delta(u-v)\,du\,dv$

$$= \frac{N_0}{2}\int_0^T s_1(u)s_2(u)\,du$$

$$= 0 \qquad \because s_1(t) \text{ and } s_2(t) \text{ are orthogonal over } (0, T)$$

$E[n_1 n_2] = 0$. So, n_1 and n_2 are also orthogonal.

Q.10 Find the time autocorrelation function of the signal $g(t) = e^{-at}\, u(t)$ and from this obtain the energy spectral density (ESD) of $g(t)$.

Solution:

Auto correlation function,

$$R_x(\tau) = \int_{-\infty}^{\infty} g(t)\, g(t-\tau)dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-a(t-\tau)}u(t-\tau)dt$$

$$= \int_{\tau}^{\infty} e^{-2at}e^{a\tau}dt = e^{a\tau}\int_{\tau}^{\infty} e^{-2at}dt = \frac{e^{a\tau}}{-2a}\left[e^{-2at}\right]_{\tau}^{\infty} = \frac{e^{a\tau}}{2a}e^{-2a\tau} = \frac{e^{-a\tau}}{2a}$$

Similar process is valid for negative side because for real $g(t)$, $R_x(\tau)$ is even function of time

$$R_x(\tau) = \frac{e^{-a|\tau|}}{2a}$$

Now we know that

$$R_x(\tau) \xrightarrow{F.T} S_x(\omega)$$

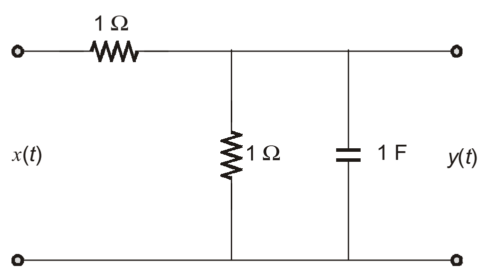
Energy spectral density

$$S_x(\omega) = \frac{1}{\omega^2 + a^2} \triangleq |G(\omega)|^2, \text{ where } G(\omega) = F.T\,[g(t)]$$

Q.11 If the input to a low-pass filter as shown below in Figure is a random process $x(t)$ with autocorrelation function, $R_x(\tau) = 5\delta(t)$, then find

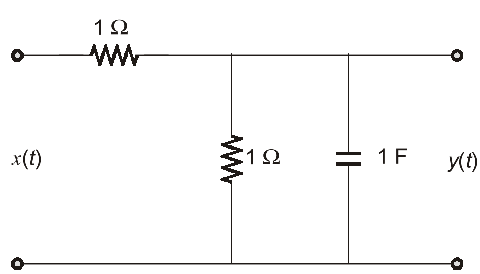
(i) Power spectral density of the output random process;

(ii) Average power of output random process.



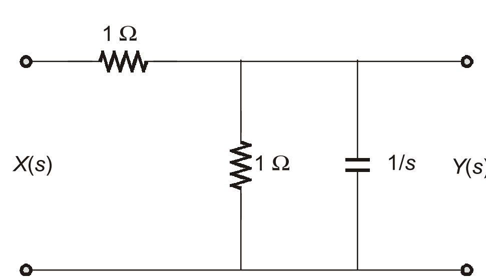
Solution:

Given low pass filter,



By taking Laplace transform

Transfer function, $H(s) = \dfrac{1}{s+2}$; $\quad H(j\omega) = \dfrac{1}{j\omega + 2}$



Input Auto-correlation function

$$R_x(\tau) = 5\delta(t)$$

Power spectral density $S_x(\omega) = F[R_x(\tau)] = 5$

Output power spectral density

$$S_y(\omega) = |H(j\omega)|^2\, S_x(\omega)$$

$$S_y(\omega) = \frac{5}{\omega^2 + 4}$$

Output Auto-correlation function = $R_y(\tau) = \frac{5}{4}e^{-2|t|}$

$$\because \quad e^{-|a|t} \xleftrightarrow{\text{CTFT}} \frac{2a}{\omega^2 + a^2}$$

(ii) Average power of the output process = $R_y(0)$

$$P_y = \frac{5}{4}\text{ W}$$

Q.12 It is desired to generate a random signal $X(t)$ with auto correlation function, $R_x(\tau) = 5\eta e^{-5|\tau|}$, by passing a white noise $n(t)$, with power spectral density $S_n(f) = \eta/2$ W/Hz, through an LTI system. Obtain an expression for the transfer function $H(f)$ of the given LTI system.

Solution:

Given that, $R_X(\tau) = 5\eta e^{-5|\tau|}$

So, the spectral density = Fourier transform of $R_x(\tau)$

$$\Rightarrow \quad S_X(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} 5\eta e^{-5|\tau|}e^{-j\omega\tau}d\tau$$

$$= 5\eta\left[\int_0^{\infty} e^{-5\tau}e^{-j\omega\tau}d\tau + \int_0^{\infty} e^{-5\tau}e^{j\omega\tau}d\tau\right]$$

$$= 5\eta\left[\frac{e^{-(5+j\omega)\tau}}{-(5+j\omega)}\right]_0^{\infty} + 5\eta\left[\frac{e^{-(5-j\omega)\tau}}{-(5-j\omega)}\right]_0^{\infty} = 5\eta\left[\frac{1}{5+j\omega} + \frac{1}{5-j\omega}\right]$$

$$\therefore \quad S_X(\omega) = \frac{5\eta \times 10}{25+\omega^2} = \frac{50\eta}{25+\omega^2} \quad \text{...(i)}$$

Hence,
$$S_X(\omega) = |H(\omega)|^2 \cdot \frac{\eta}{2}$$

$$\therefore \quad |H(\omega)|^2 = S_X(\omega)\cdot\frac{2}{\eta} = \frac{100\eta}{25+\omega^2}\cdot\frac{1}{\eta}$$

$$\Rightarrow \quad |H(f)|^2 = \frac{100}{25+4\pi^2 f^2}$$

$$\therefore \quad |H(f)| = \frac{10}{\sqrt{25+4\pi^2 f^2}} \quad \Rightarrow H(f) = \frac{10}{5+j2\pi f}$$

Q.13 The power spectral density of a real stationary random process $X(t)$ is given by

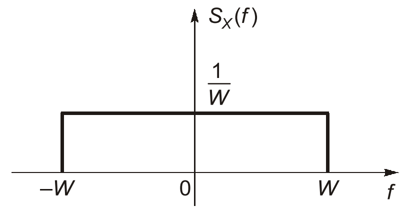
$$S_X(f) = \begin{cases} \frac{1}{W}, & |f| \le W \\ 0 & |f| > W \end{cases}$$

Find the value of the expectation $E\left[\pi X(t)\cdot X\left(t - \frac{1}{4W}\right)\right]$.

Solution:

$$S_X(f) = \begin{cases} \frac{1}{W}, & |f| \le W \\ 0 & |f| > W \end{cases}$$

$\therefore$



$\because$ $$R_X(\tau) \xleftrightarrow{\text{CTFT}} [S_X(f)]$$

$$R_X(\tau) = 2\text{sinc}[2W\tau]$$

Also,

$$E\left[\pi X(t)\cdot X\left(t-\frac{1}{4W}\right)\right] = \pi E\left[x(t)\cdot x\left(t-\frac{1}{4W}\right)\right]$$

$$= \pi R_X\left(\frac{1}{4W}\right) = \pi\cdot 2\text{sinc}\left(\frac{1}{2}\right) \quad \text{...(i)}$$

We know that, $$\text{sinc}(x) = \frac{\sin\pi x}{\pi x}$$

From equation (i), we get

$\therefore$ $$E\left[\pi X(t)\cdot X\left(t-\frac{1}{4W}\right)\right] = 2\pi\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = 4$$

Q.14 Stationary random process $X(t)$ has the following auto correlation function

$$R_X(\tau) = \sigma^2 e^{-\mu|\tau|}$$

where μ and σ^2 are constants. It is passed through a filter whose impulse response is

$$h(\tau) = \alpha e^{-\alpha\tau}u(\tau)$$

here α is constant and $u(\tau)$ is step function.

(a) Find power spectral density of random signal $X(t)$.

(b) Find the power spectral density of the output random signal $Y(t)$.

Solution:

(a) Power spectral density (PSD) of input is expressed as

$$S_X(\omega) = \text{CTFT}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}\,d\tau$$

or $$S_X(\omega) = \int_{-\infty}^{\infty}\sigma^2 e^{-\mu|\tau|}e^{-j\omega\tau}\,d\tau = \sigma^2\int_{-\infty}^{0} e^{(\mu-j\omega)\tau}\,d\tau + \sigma^2\int_{0}^{\infty} e^{-(\mu+j\omega)\tau}\,d\tau$$

$$= \sigma^2\left[\frac{1}{\mu-j\omega}+\frac{1}{\mu+j\omega}\right] = \frac{2\mu\sigma^2}{\mu^2+\omega^2}$$

(b) Power spectral density of output is related to power spectral density of input as

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) \quad \text{...(i)}$$

and $$H(e^{j\omega}) = \text{CTFT}\{h(\tau)\}$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\alpha e^{-\alpha\tau}u(\tau)e^{-j\omega\tau}\,d\tau$$