Electrical Engineering

Control Systems

Comprehensive Theory

with **Solved Examples** and **Practice Questions**

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Control Systems

Chapter 12

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CHAPTER 2

Transfer Function

2.1 Transfer Function and Impulse Response Function

In control theory, transfer functions are commonly used to characterise the input-output relationships of components or systems that can be described by linear, time-invariant differential equations.

Transfer Function

The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Linear Systems

A system is called linear if the principle of superposition and principle of homogeneity apply. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses. Hence, for the linear system, the response to several inputs can be calculated by transferring one input at a time and adding the results. It is the principle that allows one to build up complicated solutions to the linear differential equations from simple solutions.

In an experimental investigation of a dynamic system, if cause and effect are proportional, thus implying that the principle of superposition holds, then the system can be considered as linear.

Linear Time-Invariant Systems and Linear-Time Varying Systems

A differential equation is linear if the coefficients are constants or functions only of the independent variable. Dynamic systems that are composed of linear time-invariant lumped-parameter components may be described by linear time-invariant differential equations i.e. constant-coefficient differential equations. Such systems are called linear time-invariant (or linear constant-coefficient) systems. Systems that are represented by differential equations whose coefficients are function of time are called linear time varying systems. An example of a time-varying control system is a space craft control system (the mass of a space craft changes due to fuel consumption).

The definition of transfer function is easily extended to a system with multiple inputs and outputs (i.e. a multivariable system). In a multivariable system, a linear differential equation may be used to describe the relationship between a pair of input and output variables, when all other inputs are set to zero. Since the principle of superposition is valid for linear systems, the total effect (on any output) due to all the inputs acting simultaneously is obtained by adding up the outputs due to each input acting alone.

Example-2.1 When deriving the transfer function of a linear element

(a) both initial conditions and loading are taken into account
(b) initial conditions are taken into account but the element is assumed to be not loaded.
(c) initial conditions are assumed to be zero but loading is taken into account
(d) initial conditions are assumed to be zero and the element is assumed to be not loaded.

***Solution:* (c)**

While deriving the transfer function of a linear element only initial conditions are assumed to be zero, loading (or input) can't assume to be zero.

Example-2.2 If the initial conditions for a system are inherently zero, what does it physically mean?

(a) The system is at rest but stores energy
(b) The system is working but does not store energy
(c) The system is at rest or no energy is stored in any of its part
(d) The system is working with zero reference input

***Solution:* (c)**

A system with zero initial conditions is said to be at rest since there is no stored energy.

Example-2.3 What are the properties of linear systems not valid for non-linear systems? Explain each briefly?

Solution:

- Linear systems satisfy properties of superposition and homogeneity. Any system that does not satisfy these properties is non-linear.

  Property of superposition: When the output corresponding to v_{in_1} is v_{out_1} and the output corresponding to v_{in_2} is v_{out_2} then the output corresponding to $av_{in_1} + bv_{in_2}$ is $av_{out_1} + bv_{out_2}$.

  Property of homogeneity: It states that for a given input x in the domain of the function *f* and for any real number *k*

  $$f(kx) = kf(x)$$

- Linear systems have one equilibrium point at the origin. Non-linear systems may have many equilibrium points.

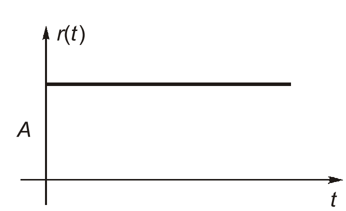
2.2 Standard Test Signals

1. Step Signal

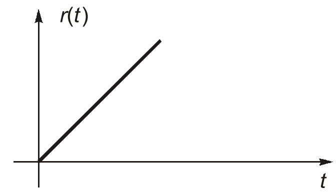
$$r(t) = A\,u(t)$$

where, $$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

Laplace transform, $$R(s) = A/s$$



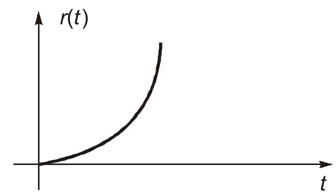
2. Ramp Signal



$$r(t) = \begin{cases} A\,t & t > 0 \\ 0 & t < 0 \end{cases}$$

Laplace transform, $R(s) = A/s^2$

3. Parabolic Signal

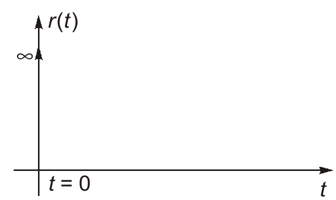


$$r(t) = \begin{cases} A\,t^2/2 & , \ t > 0 \\ 0 & , \ t < 0 \end{cases}$$

Laplace transform, $R(s) = A/s^3$

4. Impulse Signal

$$r(t) = \begin{cases} \infty & , \ t = 0 \\ 0 & , \ t \neq 0 \end{cases} \quad ; \quad \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$



Laplace transform, $R(s) = 1$

Transfer function, $G(s) = \dfrac{C(s)}{R(s)}$

$C(s) = F(s)\,R(s)$

Let, $R(s) = \text{Impulse signal} = 1$

$C(s) = \text{Impulse response} = G(s) \times 1 = \text{T.F.}$

$$\mathcal{L}\{\text{Impulse Response}\} = \text{Transfer function} = \left[\frac{C(s)}{R(s)}\right]$$

NOTE

- d/dt (Parabolic Response) = Ramp Response
- d/dt (Ramp Response) = Step Response
- d/dt (Step Response) = Impulse Response

Consider, a linear time-invariant system has the input $u(t)$ and output $y(t)$. The system can be characterized by its impulse response $g(t)$, which is defined as the output when the input is a unit-impulse function $\delta(t)$. Once the impulse response of a linear system is known, the output of the system $y(t)$, with any input $u(t)$, can be found by using the transfer function.

Let $G(s)$ denotes the transfer function of a system with input $u(t)$, output $y(t)$, and impulse response $g(t)$. The transfer function $G(s)$ is defined as

$$G(s) = \mathcal{L}[g(t)] = \left.\frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]}\right|_{\text{initial conditions}\to 0} = \frac{Y(s)}{U(s)}$$

Remember

Sometimes, students do a common mistake, they first find $y(t)/u(t)$ and then take its Laplace transform to determine the transfer function which is absolutely wrong. Because,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]} \neq \mathcal{L}\left[\frac{y(t)}{u(t)}\right]$$

2.3 Poles and Zeros of a Transfer Function

The transfer function of a linear control system can be expressed as

$$G(s) = \frac{A(s)}{B(s)} = \frac{K(s-s_1)(s-s_2)\dots(s-s_n)}{(s-s_a)(s-s_b)\dots(s-s_m)}$$

where K is known as gain factor of the transfer function $G(s)$.

In the transfer function expression, if s is put equal to s_a, s_b ... s_m then it is noted that the value of the transfer function is infinite. These s_a, s_b, ... s_m are called the poles of the transfer function.

In the transfer function expression, if s is put equal to s_1, s_2 ... s_n then it is noted that the value of the transfer function is zero. These s_1, s_2 ... s_n are called the zeros of the transfer function.

Multiple Poles and Multiple Zeros

The poles s_a, s_b ... s_m or the zeros s_1, s_2 ... s_n are either real or complex and the complex poles or zeros always appear in conjugate pairs.

It is possible that either poles or zeros may coincide; such poles or zeros are called multiple poles or multiple zeros.

Simple Poles and Simple Zeros

Non-coinciding poles or zeros are called simple poles or simple zeros. From the transfer function expression, it is observed that

- If $n > m$, then the value of transfer function is found to be infinity for $s = \infty$, Hence, it is concluded that there exists a pole of the transfer function at infinity (∞) and the multiplicity (order) of such a pole being $(n-m)$.
- If $n < m$, then the value of transfer function is found to be zero for $s = \infty$. Hence, it is concluded that there exists a zero of the transfer function at infinity (∞) and the multiplicity (order) of such a zero being $(m-n)$.

  Therefore, for a rational transfer function the total number of zeros is equal to the total number of poles.

  The transfer function of a system is completely specified in terms of its poles, zeros and the gain factor.

Consider the following transfer function:

$$G(s) = \frac{s+3}{(s+2)(s+1+3j)(s+1-3j)}$$

For the above transfer function, the poles are at

(a) $s_a = -2$ (b) $s_b = -1 - 3j$ and (c) $s_c = -1 + 3j$

The zeros are at $s_1 = -3$.

As the number of zeros should be equal to number of poles, the remaining two zeros are located at $s = \infty$.

The pole-zero plot is plotted as shown:

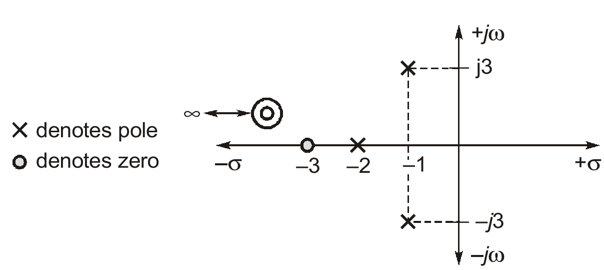


Figure-2.1 : *Pole-zero plot*

Poles and zero are those complex/critical frequencies which make the transfer function infinity or zero.

Proper Transfer Functions

The transfer functions are said to be strictly proper if the order of the denominator polynomial is greater than that of the numerator polynomial (i.e. $m > n$). If $m = n$, the transfer function is called proper. The transfer function is improper if $n > m$.

In the transfer function expression of a control system, the highest power of s in the numerator is generally either equal to or less than that of the denominator.

Example-2.4 A transfer function has two zeros at infinity. Then the relation between the numerator degree (N) and the denominator degree (M) of the transfer function is

(a) $N = M + 2$ (b) $N = M - 2$
(c) $N = M + 1$ (d) $N = M - 1$

***Solution:* (b)**

For a rational transfer function, the total number of zeros are equal to total number of poles.

Therefore, Number of poles $= M$; Number of zeros $= N + 2$

For a rational transfer function: $M = N + 2$ or $N = M - 2$

2.4 Properties of Transfer Function

The properties of the transfer function are summarized as follows:

1. The transfer function is defined only for a linear time-invariant system. It is not defined for non-linear or time variant systems.
2. The transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response. Alternately, the transfer function between a pair of input and output variables is the ratio of the Laplace transform of the output to the Laplace transform of the input.
3. All initial conditions of the system are set to zero.
4. Transfer function is independent of the input of the system.
5. The transfer function of a continuous-data system is expressed only as a function of the complex variables. It is not a function of the real variable, time, or any other variable that is used as the independent variable or discrete-data system modelled by difference equations, the transfer function is a function of Z, when the Z-transform is used.
6. If the system transfer function has no poles or zeros with positive real parts, the system is a minimum phase system.
   Non-minimum phase functions are the functions which have poles or zeros on right hand side of s-plane.
7. The stability of a time-invariant linear system can be determined from its characteristic equation.
   Characteristic equation: The characteristic equation of a linear system is defined as the equation obtained by setting the denominator polynomial of the closed loop transfer function to zero.

Example-2.5 State and explain minimum phase and non-minimum phase transfer functions with examples.

Solution:

Minimum phase transfer function:

⇒ Transfer functions which have all poles and zeros in the left half of the s-plane i.e. system having no poles and zeros in the RHS of the s-plane are minimum phase transfer functions.

⇒ On the other hand, a transfer function which has one or more zeros in the right half of s-plane is known as "**non-minimum phase transfer function**".

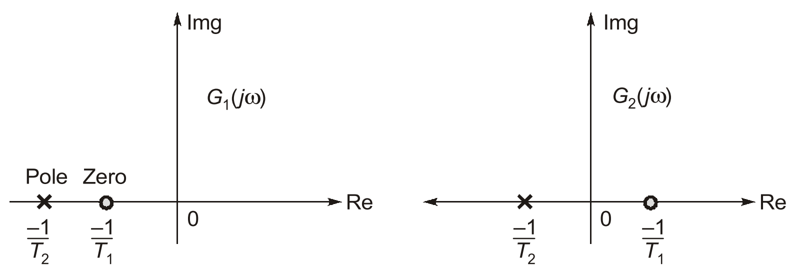
Let $$G_1(s) = \frac{1+sT_1}{1+sT_2}$$

⇒ $$G_1(j\omega) = \frac{1+j\omega T_1}{1+j\omega T_2} \quad \text{...(i)}$$

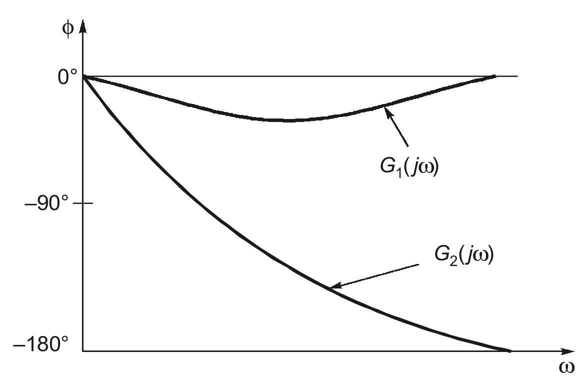
and $$G_2(j\omega) = \frac{1-j\omega T_1}{1+j\omega T_2} \quad \text{...(ii)}$$

The transfer function given by equation (i) represents the minimum-phase transfer function and equation (ii) represents the non-minimum phase transfer function.

⇒ The pole-zero configuration of above transfer function as given by equation (i) and (ii) may be drawn as:



⇒ The ***minimum phase function*** has unique relationship between its phase and magnitude curves. Typical phase angle characteristics are shown below:



⇒ It will be seen that larger the phase lags present in a system, the more complex are its stabilization problems. Therefore, in control systems, elements with non minimum phase transfer function are avoided as far as possible.

⇒ A common example of a non-minimum phase system is "***transportation lag***" which has the transfer function,

$$G(j\omega) = e^{-j\omega T} = 1\angle -\omega T \text{ Radian}$$

$$= 1\angle -57.3\,\omega T \text{ degree}$$

2.5 Methods of Analysis

Methods of analysis of a system involves:

(*a*) Transfer function approach

(*b*) State variable approach

Many a times in interviews the relative comparison of these two approaches has been asked, which we will understand during the study of state variable analysis (Refer "*introduction*" of chapter 12).

Advantages of Transfer Function Approach

1. It gives simple mathematical algebraic equation.
2. It gives poles and zeros of the system directly.
3. Stability of the system can be determined easily.
4. The output of the system for any input can be determined easily.

Disadvantages of Transfer Function Approach

1. It is applicable only for LTI system.
2. It does not take initial conditions into account.
3. The internal states of the system can not be determined.
4. Analysis of multiple input multiple output systems is cumbersome.
5. Controllability and observability can not be determined.

There are two basic control loop configurations:

(*a*) **Closed loop control system:** In this configuration, the changes in the output are measured through feedback and compared with input to achieve the control objective.

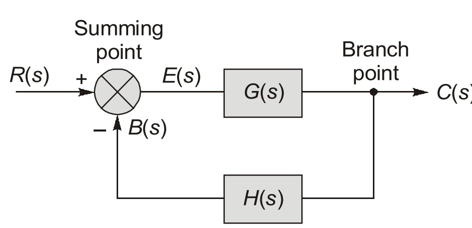


Figure-2.2

$$E(s) = R(s) - B(s)$$

$$\frac{C(s)}{G(s)} = R(s) - C(s)\,H(s)$$

$$\Rightarrow \quad C(s)\,[1 + G(s)\,H(s)] = G(s)\,R(s)$$

$$\Rightarrow \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

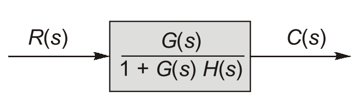
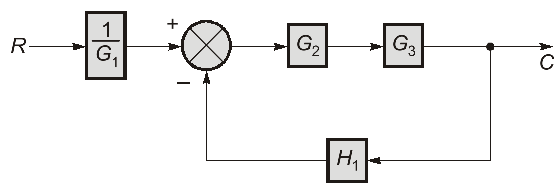


Figure-2.3

Hence, closed loop transfer function [C.L.T.F.] = $\dfrac{G(s)}{1+G(s)\,H(s)}$

Student's Assignments 1

Q.1 A feedback control system is shown below. Find the transfer function for this system.

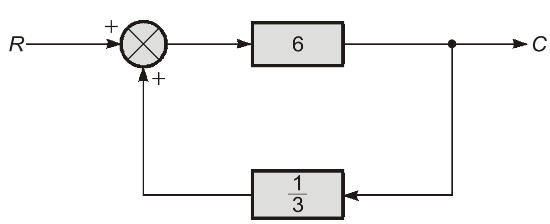


Q.2 The step response of a system is given as

$y = 1 - \frac{7}{3}e^{-t} + \frac{3}{2}e^{-2t} - \frac{1}{6}e^{-4t}$. If the transfer function of this system is $\frac{(s+a)}{(s+b)(s+c)(s+d)}$ then $a + b + c + d$ is __________.

Q.3 A system has the transfer function $\frac{(1-s)}{(1+s)}$. Its gain at ω = 1 rad/sec is ______.

Q.4 The close loop gain of the system shown below is



Student's Assignments 1 Explanations

1. $\left(\frac{G_2G_3}{G_1(1+H_1G_2G_3)}\right)$

Multiply G_2 and G_3 and apply feedback formula and then again multiply with $\frac{1}{G_1}$

$$T(s) = \frac{G_2G_3}{G_1(1+G_2G_3H_1)}$$

2. (15)

$$y = 1 - \frac{7}{3}e^{-t} + \frac{3}{2}e^{-2t} - \frac{1}{6}e^{-4t}$$

$$p(t) = \frac{dy}{dt}$$

$$= \frac{7}{3}e^{-t} + \frac{3}{2}\times(-2)\times e^{-2t} - \left(\frac{1}{6}\right)(-4)e^{-4t}$$

Laplace transform of p(t)

$$p(s) = \frac{7/3}{s+1} + \frac{-3}{s+2} + \frac{2/3}{s+4}$$

$$= \frac{s+8}{(s+1)(s+2)(s+4)}$$

$\Rightarrow \quad a + b + c + d = 15$

3. (1)

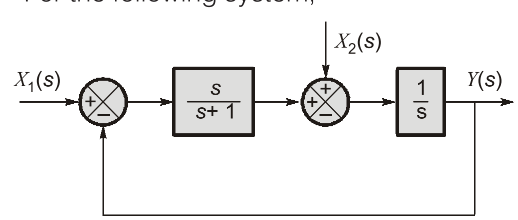
For all pass system, gain = '1' at all frequencies.

4. (–6)

$$\text{C.L.T.F.} = \frac{6}{1-6\times\frac{1}{3}} = \frac{6}{-1} = -6$$

Student's Assignments 2

Q.1 For the following system,

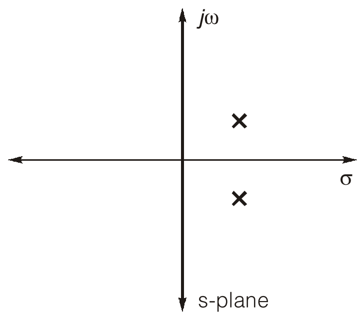


when $X_1(s)$ = 0, the transfer function $\frac{Y(s)}{X_2(s)}$ is

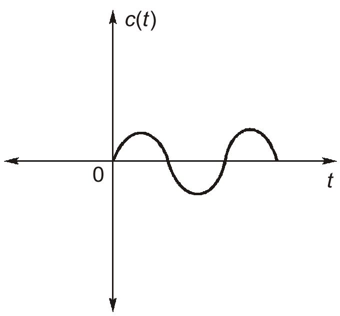
(a) $\frac{s+1}{s^2}$ (b) $\frac{1}{s+1}$

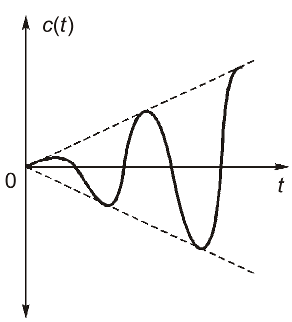
(c) $\frac{s+2}{s(s+1)}$ (d) $\frac{s+1}{s(s+2)}$

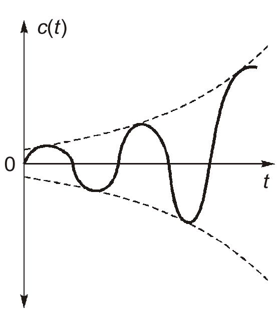
Q.2 If closed-loop transfer function poles shown below

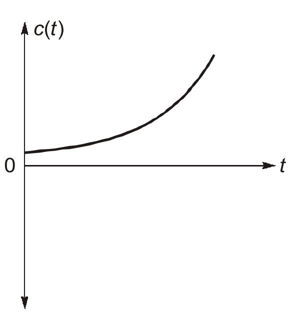


Impulse response is

(a)


(b)


(c)


(d)


Q.3 The impulse response of several continuous systems are given below. Which is/are stable?

1. $h(t) = te^{-t}$
2. $h(t) = 1$
3. $h(t) = e^{-t} \sin 3t$
4. $h(t) = \sin \omega t$

(a) 1 only (b) 1 and 3
(c) 3 and 4 (d) 2 and 4

Q.4 Ramp response of the transfer function

$$F(s) = \frac{s+1}{s+2} \text{ is}$$

(a) $\frac{1}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}t$ (b) $\frac{1}{4}e^{-2t} + \frac{1}{4} + \frac{1}{2}t$

(c) $\frac{1}{2} - \frac{1}{2}e^{-2t} + t$ (d) $\frac{1}{2}e^{-2t} + \frac{1}{2} - t$

Q.5 Which of the following statements are correct?

1. Transfer function can be obtained from the signal flow graph of the system.
2. Transfer function typically characterizes to linear time invariant systems.
3. Transfer function gives the ratio of output to input in frequency domain of the system.

(a) 1 and 2 (b) 2 and 3
(c) 1 and 3 (d) 1, 2 and 3

Q.6 Which of the following is not a desirable feature of a modern control system?

(a) Quick response
(b) Accuracy
(c) Correct power level
(d) Oscillations

Q.7 In regenerating feedback, the transfer function is given by

(a) $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

(b) $\frac{C(s)}{R(s)} = \frac{G(s)\,H(s)}{1-G(s)\,H(s)}$

(c) $\frac{C(s)}{R(s)} = \frac{G(s)H(s)}{1+G(s)H(s)}$

(d) $\frac{C(s)}{R(s)} = \frac{G(s)}{1-G(s)H(s)}$

Q.8 In a continuous data system
(a) data may be a continuous function of time at all points in the system
(b) data is necessarily a continuous function of time at all points in the system
(c) data is continuous at the input and output parts of the system but not necessarily during intermediate processing of the data
(d) only the reference signal is a continuous function of time

Q.9 The principle of homogeneity and superposition are applied to
(a) linear time variant systems
(b) non-linear time variant systems
(c) linear time invariant systems
(d) non-linear time invariant systems

Q.10 Consider the following statements regarding the advantages of closed loop negative feedback control systems over open-loop systems:
1. The overall reliability of the closed loop systems is more than that of open-loop system.
2. The transient response in the closed loop system decays more quickly than in open-loop system.
3. In an open-loop system, closing of the loop increases the overall gain of the system.
4. In the closed-loop system, the effect of variation of component parameters on its performance is reduced.

Of these statements
(a) 1 and 3 are correct
(b) 1, 2 and 4 are correct
(c) 2 and 4 are correct
(d) 3 and 4 are correct

Q.11 Match **List-I** (Time function) with **List-II** (Laplace transforms) and select the correct answer using the codes given below lists:

List-I	List-II
A. $[af_1(t) + bf_2(t)]$	1. $aF_1(s) + bF_2(s)$
B. $[e^{-at}f(t)]$	2. $sF(s) + f(0)$
C. $\left[\frac{df(t)}{dt}\right]$	3. $\frac{1}{s}F(s)$
D. $\left[\int_0^t f(x)dx\right]$	4. $sF(s) - f(0^-)$
	5. $F(s + a)$

Codes:

	A	B	C	D
(a)	5	2	3	4
(b)	1	5	4	3
(c)	2	1	3	4
(d)	1	5	3	4

Q.12 If a system is represented by the differential equation, is of the form $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = r(t)$
(a) $k_1\ e^{-t} + k_2 e^{-9t}$
(b) $(k_1 + k_2)\ e^{-3t}$
(c) $ke^{-3t}\sin(t + \phi)$
(d) $te^{-3t}\ u(t)$

Q.13 A linear system initially at rest, is subject to an input signal $r(t) = 1 - e^{-t}\ (t \geq 0)$. The response of the system for $t > 0$ is given by $c(t) = 1 - e^{-2t}$. The transfer function of the system is
(a) $\frac{(s+2)}{(s+1)}$
(b) $\frac{(s+1)}{(s+2)}$
(c) $\frac{2(s+1)}{(s+2)}$
(d) $\frac{(s+1)}{2(s+2)}$

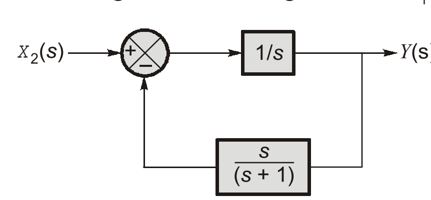
Answer Key:

1. (d)	**2.** (c)	**3.** (b)	**4.** (a)	**5.** (d)
6. (d)	**7.** (d)	**8.** (b)	**9.** (c)	**10.** (b)
11. (b)	**12.** (d)	**13.** (c)		

Student's Assignments 2 Explanations

1. (d)

Redrawing the block diagram with $X_1(s) = 0$



The transfer function

$$T(s) = \frac{Y(s)}{X_2(s)} = \frac{G(s)}{1+G(s)H(s)} \quad \text{...(i)}$$

Here, $G(s) = \frac{1}{s}$ and $H(s) = \frac{s}{s+1}$

$$\frac{Y(s)}{X_2(s)} = \frac{1/s}{1+\frac{1}{s}\times\frac{s}{s+1}} = \frac{(s+1)}{s(s+2)}$$

2. (c)

$$\text{T.F.} = \frac{1}{[s-(\sigma+j\omega)][s-(\sigma-j\omega)]}$$

$$= \frac{1}{[(s-\sigma)-j\omega)][(s-\sigma)+j\omega)]}$$

$$= \frac{1}{\left[(s-\sigma)^2-(j\omega)^2\right]} = \frac{1}{\left[(s-\sigma)^2+\omega^2\right]}$$

For impulse response, taking its inverse Laplace transformation we get,

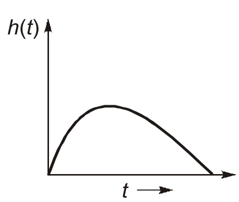
$$c(t) = e^{\sigma t}\sin\omega t$$

hence option (c) is correct.

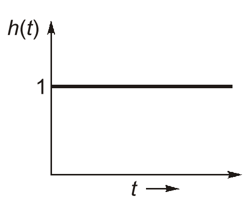
3. (b)

If the impulse response decays to zero as time approaches infinity, the system is stable.

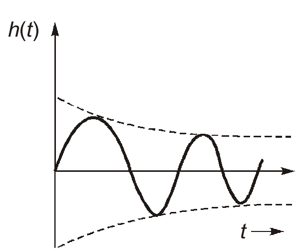
(1)



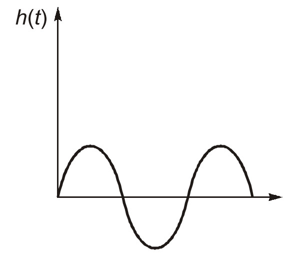
(2)



(3)



(4)



4. (a)

$$\frac{C(s)}{R(s)} = \frac{s+1}{s+2}$$

$$\therefore \quad C(s) = R(s)\cdot\frac{s+1}{s+2}$$

$$= \frac{1}{s^2}\cdot\frac{s+1}{s+2} = \frac{1}{s^2}\left(1-\frac{1}{s+2}\right)$$

$$= \frac{1}{s^2} - \frac{1}{s^2(s+2)}$$

$$\frac{1}{s^2}\cdot\left(\frac{s+1}{s+2}\right) = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}$$

$$s+1 = As(s+2)+B(s+2)+Cs^2$$

$$= As^2+2As+Bs+2B+Cs^2$$

$\therefore$ $A + C = 0$, $2A + B = 1$ and $2B = 1$

$$\therefore \quad A = \frac{1}{2}, \quad B = \frac{1}{4}, \quad C = -\frac{1}{4}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{1}{4s}+\frac{1}{2s^2}+\left(-\frac{1}{4}\right)\frac{1}{s+2}$$

$$= \frac{1}{4}u(t)+\frac{1}{2}tu(t)-\frac{1}{4}e^{-2t}u(t)$$

5. (d)

(i) Transfer function can be obtained from signal flow graph of the system.

(ii) Transfer function typically characterizes to LTI systems.

(iii) Transfer function gives the ratio of output to input in s-domain of system.

$$\text{TF} = \left.\frac{L[\text{Output}]}{L[\text{Input}]}\right|_{\text{Initial conditions} = 0}$$