COMPUTER SCIENCE & INFORMATION TECHNOLOGY

Discrete and Engg. Mathematics

Comprehensive Theory
with Solved Examples *and* Practice Questions

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Discrete & Engg. Mathematics

EDITIONS

First Edition : 2015
Second Edition : 2016
Third Edition : 2017
Fourth Edition : 2018
Fifth Edition : 2019
Sixth Edition : 2020
Seventh Edition : 2021
Eighth Edition : 2022
Ninth Edition : 2023

Tenth Edition : 2024

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Discrete & Engg. Mathematics

Discrete & Engineering Mathematics

Goal of the Subject

The mathematics of modern computer science is built almost entirely on discrete math, in particular Combinatorics and graph theory. Discrete math will help you with the "Algorithms, Complexity and Computability Theory" part of the focus more than programming language. The understanding of set theory, probability, matrices and combinations will allow us to analyze algorithms. We will be able to successfully identify parameters and limitations of your algorithms and have the ability to realize how complex a problem/solution is.

Discrete & Engineering Mathematics

INTRODUCTION

In this book we tried to keep the syllabus of Discrete & Engineering Mathematics around the GATE syllabus. Each topic required for GATE is crisply covered with illustrative examples and each chapter is provided with Student Assignment at the end of each chapter so that the students get a thorough revision of the topics that he/she had studied. This subject is carefully divided into eight chapters as described below.

Discrete Mathematics:

1. **Propositional Logic**: In this chapter we study logical connectives, well-Formed formulas, rules for inference, predicate calculus with Universal and Existential quantifiers.
2. **Combinatorics**: In this chapter we discuss the basic principles of counting, permutations, combinations, generating functions, binomial coefficients, summations and finally we discuss the recurrence relations.
3. **Set Theory and Algebra**: In this chapter we discuss the basic terms and definitions of set theory, Operations on sets, relations and types of relations , functions and their types and finally group theory, posets, lattices and boolean algebra.
4. **Graph Theory**: In this chapter we discuss the Special Graphs, isomorphism, vertex and edge connectivity, Euler graphs, Hamiltonian and planar graph, trees and enumeration of graphs.

Engineering Mathematics:

5. **Probability:** In this chapter we discuss the basic probability and axioms of probability, Basic concepts of statistics (mean, mode, variance and standard deviation), Discrete and continuous random variables and their distributions.
6. **Linear Algebra:** In this chapter we discuss the Special matrices, Algebra of matrices and their properties, inverse of a matrix, determinant of a matrix, solution of system of linear equations, LU decomposition method, Eigen values and Eigen vectors and finally we discuss the Cayley Hamilton theorem.
7. **Calculus**: In this chapter we discuss about Limits, continuity and differentiability, differentiation, partial derivatives, applications of differentiation (Mean value theorems, increasing and decreasing functions and maxima and minima of functions), methods of integration, and finally definite and indefinite integrals and their properties.

■■■■

Propositional Logic

CHAPTER

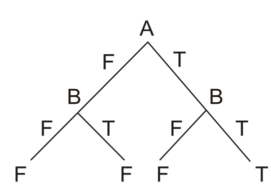
1

1.1 PROPOSITIONAL LOGIC; FIRST ORDER LOGIC

Logic: In general logic is about reasoning. It is about the validity of arguments. Consistency among statements and matters of truth and falsehood. In a formal sense logic is concerned only with the form of arguments and the principle of valid inferencing. It deals with the notion of truth in an abstract sense.

Truth Tables: Logic is mainly concerned with valid deductions. The basic ingredients of logic are logical connectives, and, or, not , if.... then ...., if and only if etc. We are concerned with expressions involving these connectives. We want to know how the truth of a compound sentence like, "x = 1 and y = 2" is affected by, or determined by, the truth of the separate simple sentences "x = 1", "y = 2".

Truth tables present an exhaustive enumeration of the truth values of the component propositions of a logical expression, as a function of the truth values of the simple propositions contained in them. An example of a truth table is shown in table 1 below. The information embodied in them can also be usefully presented in tree form.



The branches descending from the node A are labelled with the two possible truth values for A. The branches emerging from the nodes marked B give the two possible values for B for each value of A. The leaf nodes at the bottom of the tree are marked with the values of A $\wedge$ B for each truth combination of A and B.

1.2 LOGICAL CONNECTIVES OR OPERATORS

The following symbols are used to represent the logical connectives or operators.

And	$\wedge$(Conjuction)
or	$\vee$(Disjunction)
not	$\neg$(Not)
Ex - or	$\oplus$
Nand	$\uparrow$
Nor	$\downarrow$
if....then	$\rightarrow$(Implication)
if and only if	$\leftrightarrow$(Biconditional)

1. **$\wedge$ (And / Conjuction):** We use the letters F and T to stand for false and true respectively.

Table-1

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

It tells us that the conjuctive operation $\wedge$ is being treated as a **binary** logical connective–it operates on two logical statements. The letters A and B are **"Propositional Variables"**.

The table tells us that the compound proposition $A \wedge B$ is true only when both A and B are true separately. The truth table tells us how to do this for the operator. $A \wedge B$ is called a truth function of A and B as its value is dependent on and determined by the truth values of A and B.

A and B can be made to stand for the truth values of propositions as follows:

A : The cat sat on the mat

B : The dog barked

Each of which may be true or false. Then $A \wedge B$ would represent the compound proposition "The cat sat on the mat and the dog barked"

$A \wedge B$ is written as A.B in Boolean Algebra.

2. **$\vee$ (Disjunction):** The truth table for the disjunctive binary operation $\vee$ tells us that the compound proposition $A \vee B$ is false only if A and B are both false, otherwise it is true.

A	B	$A \vee B$
F	F	F
F	T	T
T	F	T
T	T	T

This is inclusive use of the operator 'or'.

In Boolean Algebra $A \vee B$ is written as A + B.

3. **$\neg$ (Not):**

The negation operator is a **"unary operator"** rather than a binary operator like $\wedge$ and its truth table is

A	$\neg A$
F	T
T	F

The table presents ¬ in its role i.e., the negation of true is false, and the negation of false is true. Notice that ¬ A is sometimes written as ~A or A′.

4. **⊕ (Exclusive OR or Ex - OR):**

A ⊕ B is true only when either A or B is true but not when both are true or when both are false. A ⊕ B is also denoted by A ∨ – B.

A	B	A ⊕ B
F	F	F
F	T	T
T	F	T
T	T	F

5. **↑ (NAND):**

$P \uparrow Q \equiv \neg(P \wedge Q)$

6. **↓ (NOR):**

$P \downarrow Q \equiv \neg(P \vee Q)$

Note:

$$P \uparrow P \equiv \neg P$$
$$P \downarrow P \equiv \neg P$$
$$(P \downarrow Q) \downarrow (P \downarrow Q) \equiv P \vee Q$$
$$(P \uparrow Q) \uparrow (P \uparrow Q) \equiv P \wedge Q$$
$$(P \uparrow P) \uparrow (Q \uparrow Q) \equiv P \vee Q$$

7. **→ (Implication):**

A	B	A → B
F	F	T
F	T	T
T	F	F
T	T	T

Note that A → B is false only when A is true and B is false. Also, note that A → B is true, whenever A is false, irrespective of the truth value of B.

8. **↔ (if and only if):** The truth table is

A	B	A ↔ B
F	F	T
F	T	F
T	F	F
T	T	T

Note that Bi-conditional (if and only if) is true only when both A & B have the same truth values. (A ↔ B may be written as $A \rightleftharpoons B$)

Equivalences: B ∧ A always takes on the same truth value is as A ∧ B.

We say that B ∧ A is logically equivalent to A ∧ B and we can write this as follows B ∧ A ≡ A ∧ B

Definition: Two expression are logically equivalent if each one always has the same truth value as the other.

Also,

$$B \wedge A \equiv A \wedge B$$
$$B \vee A \equiv A \vee B$$
$$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$$
$$A \vee (B \vee C) \equiv (A \vee B) \vee C$$

These equivalence reveal $\wedge$ and $\vee$ to be commutative and associative operations. But these are not the only important equivalences that hold between logical forms.

A	B	$A \to B$
F	F	T
F	T	T
T	F	F
T	T	T

A	B	$\neg A$	$\neg A \vee B$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

In the last columns in tables (Shown Bracketed) we have exactly same sequences of truth values. So, $A \to B \equiv \neg A \vee B$. Thus, we could do without the operation $\to$.

Now, consider the following two truth tables.

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$\neg(\neg A \vee \neg B)$
F	F	T	T	T	F
F	T	T	F	T	F
T	F	F	T	T	F
T	T	F	F	F	T

We see from the bracketed truth values that $A \wedge B$ is logically equivalent to $\neg(\neg A \vee \neg B)$. Thus $\wedge$ could be replaced by a combination of $\neg$ and $\vee$.

Similarly we could show that $\leftrightarrow$ can be replaced by a combination of $\neg$ and $\vee$.

We say therefore that $(\neg, \vee)$ forms a functionally complete set of connectives.

We can also show that $(\neg, \wedge)$ also form a functionally complete set of connectives.

The NAND operator $(\uparrow)$ by itself is also a functionally complete set. So is the Nor operator $(\downarrow)$. These both are minimal functionally complete set.

Notice that $(\vee, \wedge)$ is not a functionally complete set. Neither is $(\neg)$, $(\vee)$ or $(\wedge)$ by themselves functionally complete.

Example 1.1 Obtain the truth table for $\alpha = (P \vee Q) \wedge (P \to Q) \wedge (Q \to P)$

Solution:

P	Q	$P \vee Q$	$P \to Q$	$(P \vee Q) \wedge (P \to Q)$	$Q \to P$	α
T	T	T	T	T	T	T
T	F	T	F	F	T	F
F	T	T	T	T	F	F
F	F	F	T	F	T	F

Summary

- Two expression are logically equivalent if each one always has the same truth value as the other.
- ⊕ (EX – OR) is commutative and associative, (NAND) and (NOR) are both commutative but not associative. $P \wedge (Q \oplus R) \equiv (P \wedge Q) \oplus (P \wedge R)$
- A wff is not a proposition, but if we substitute the proposition in place of propositional variable, we get a proposition e.g., $(\neg P \wedge Q) \leftrightarrow Q$ is a wff.
- When it is not clear whether a given formula is tautology, we can construct a truth table and verify that the truth value is T for all possible combinations of truth value of the propositional variables appearing in given formula.
- A **contradiction** (or absurdity) is a wff whose truth value is F for all possible assignments of truth values to the propositional variables.
- A **contingency** is a wff which is neither a tautology nor a contradiction. In other words, a contingency is a wff which is sometimes true of sometimes false.
- Two wff a and b in propositional variables P_1, P_2, ...P_n are **equivalent** if the formula $a \leftrightarrow b$ is a tautology.
- For a given wff the PDNF form is unique the PCNF form is unique, if PDNF form or PCNF of 2 wffs are same, they are equivalent.
- Quantified parts of predicate formula such as $\forall x\, P(x)$ or $\exists x\, P(x)$ are propositions. We can assign values from the universe of discourse only to free variables in a predicate formula α.

Student's Assignments

Q.1 The logical expression $((P \wedge Q) \Rightarrow (R' \wedge P)) \Rightarrow P$
(a) a tautology
(b) a contradiction
(c) a contingency
(d) All the above

Q.2 The principal conjunctive normal form is
(a) sum of products
(b) product of sums
(c) sum of max-terms
(d) product of max-terms

Q.3 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I
A. Associative law
B. Absorption law
C. Demorgans law
D. Commutative

List-II
1. $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
2. $P \vee Q \equiv Q \vee P$
3. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
4. $P \vee (P \wedge Q) \equiv P$

Codes:

	A	B	C	D
(a)	1	2	3	4
(b)	4	3	1	2
(c)	1	4	3	2
(d)	2	1	4	2

Q.4 Consider the following statements:
S_1 : $R \vee (P \vee Q)$
is a valid conclusion from the premises
$P \vee Q$, $R \rightarrow Q$, $M \rightarrow P$ and $\neg M$
S_2 : $a \rightarrow b$, $\neg (f \vee c) \Rightarrow \neg b$
then
(a) S_1 is true and S_2 is invalid
(b) S_1 is false and S_2 is invalid
(c) Both are true
(d) Both are false

Q.5 The following propositional statement is

$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r)]$

(a) tautology
(b) contradiction
(c) neither tautology nor contradiction
(d) not decidable

Q.6 Identify the correct translation into logical notation of the following assertion
"All connected bipartite graphs are nonplanar"

(a) $\forall x \left[\begin{array}{l} \sim \text{connected}(x) \vee \sim \text{biparite} \\ (x) \wedge \sim \text{planar}(x) \end{array} \right]$

(b) $\forall x \left[\begin{array}{l} \sim \text{connected}(x) \vee \sim \text{biparite} \\ (x) \vee \sim \text{planar}(x) \end{array} \right]$

(c) $\forall x \left[\begin{array}{l} \sim \text{connected}(x) \wedge \sim \text{biparite} \\ (x) \wedge \sim \text{planar}(x) \end{array} \right]$

(d) $\forall x \left[\begin{array}{l} \sim \text{connected}(x) \wedge \sim \text{biparite} \\ (x) \vee \sim \text{planar}(x) \end{array} \right]$

Q.7 Which of the following statements are true?

(*i*) It is not possible for the propositions $P \vee Q$ and $\neg P \vee \neg Q$ to be both false. To be both false.
(*ii*) It is possible for the proposition $P \to (\neg P \to Q)$ to be false.

(a) Only (*i*) is true
(b) Only (*ii*) is true
(c) Both (*i*) and (*ii*) are true
(d) Both (*i*) and (*ii*) are false

Q.8 Which of the following statements are true?

(*i*) $((P \to Q) \to R) \to ((R \to Q) \to P)$ is a tautology
(*ii*) Let A, B be finite sets, with $|A| = m$ and $|B| = n$. The number of distinct functions $f: A \to B$ is there from A to B is m^n.

(a) Only (*i*) is true
(b) Only (*ii*) is true
(c) Both (*i*) and (*ii*) are true
(d) Both (*i*) and (*ii*) are false

Q.9 State whether the following statements are true or false?

(*i*) $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$ always holds, for all proposition P, Q.
(*ii*) $((P \vee Q) \Rightarrow Q) \Rightarrow (Q \Rightarrow (P \vee Q))$ always holds, for all propositions P.

(a) (*i*) is true, (*ii*) is false
(b) Both (*i*) and (*ii*) are true
(c) (*i*) is false, (*ii*) is true
(d) Both (*i*) and (*ii*) are false

Q.10 Which of the following is tautology?

(a) $x \vee y \to y \wedge z$
(b) $x \wedge y \to y \vee z$
(c) $x \vee y \to y \to z$
(d) $x \to y \to (y \to z)$

Q.11 Suppose

$P(x)$: x is a person.
$F(x, y)$: x is the father of y.
$M(x, y)$: x is mother of y.

What does the following indicates

$(\exists z)\ (P(z) \wedge F(x, z) \wedge M(z, y))$

(a) x is father of mother of y
(b) y is father of mother of x
(c) x is father of y
(d) None of the above

Q.12 Give the converse of "If it is raining then I get wet".

(a) If it is not raining then I get wet
(b) If it is not raining then I do not get wet
(c) If it get wet then it is raining
(d) If I do not get wet then it is not raining

Q.13 Which of the following is true?

(a) $\neg(p \Rightarrow q) \equiv p \wedge \neg q$
(b) $\neg(p \leftrightarrow q) \equiv ((p \vee \neg q) \vee (q \wedge \neg p))$
(c) $\neg(\exists x\ (p\ (x) \Rightarrow q(x))) \equiv \forall x\ (p(x) \Rightarrow q(x))$
(d) $\exists x\ p(x) \equiv \forall x\ p(x)$

Answer Key:

1. (c)	**2.** (d)	**3.** (c)	**4.** (a)	**5.** (a)
6. (b)	**7.** (a)	**8.** (d)	**9.** (c)	**10.** (b)
11. (a)	**12.** (c)	**13.** (a)		

Student's Assignments Explanations

1. (c)

The logical expression

$((P \wedge Q) \Rightarrow (R' \wedge P)) \Rightarrow P$

can be converted in Boolean Algebra notation as,

$(pq \Rightarrow r'\,p) \Rightarrow p$

$\equiv ((pq)' + r'\,p) \Rightarrow p$

$\equiv (p' + q' + r'\,p) \Rightarrow p$

$\equiv ((p' + r'p) + q') \Rightarrow p$

$\equiv ((p' + p) \cdot (p' + r') + q') \Rightarrow p$

$\equiv (p' + r' + q') \Rightarrow p$

$\equiv (p' + r' + q')' + p \equiv prq + p$

$\equiv p$

$\therefore$ The given expression is a contingency.

4. (a)

S_1 : $P \vee Q,\ R \rightarrow Q,\ M \rightarrow P,\ \sim M \Rightarrow R \vee (P \vee Q)$

In boolean algebra notation the above expression is written as

$(p + q) \cdot (q + r') \cdot (p + m') \cdot m' \Rightarrow r + p + q$

$\equiv (q + pr')\,(m') \Rightarrow r + p + q$

$\equiv qm' + pr'm' \Rightarrow r + p + q$

$\equiv (qm' + pr'm') + r + p + q$

$\equiv (q' + m)\,(p' + r + m)\ r + p + q$

$\equiv q'p' + q'r + q'm + mp' + mr + m + r + p + q$

(by absorption law) $\equiv q'p' + r + m + p + q$

$\equiv (p + p') \cdot (p + q') + r + m + q$

$\equiv p + q' + r + m + q$

$\equiv p + r + m + 1 \equiv 1$

$\therefore$ S_1 is true

S_2 : $a \rightarrow b,\ \neg(f \vee c) \Rightarrow \neg b$

In boolean Algebra notation

$S_2 \equiv (a \rightarrow b)\cdot(f \vee c)' \Rightarrow b'$

$\equiv (a' + b) \cdot (f'c') \Rightarrow b'$

$\equiv [(a' + b) \cdot (f'c')] + b'$

$\equiv (a' + b)' + (f'c')^1 + b'$

$\equiv ab' + f + c + b'$

$\equiv f + c + b'$

which is a contingency

$\therefore$ S_2 is invalid.

5. (a)

$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$

$\equiv (p' + r)\,(q' + r) \rightarrow (p + q)' + r$

$\equiv (r + p'q')' \rightarrow (p + q)' + r$

$\equiv (r + p'q')' + (p + q)' + r$

$\equiv r'(p'q')' + p'q' + r$

$\equiv r'(p + q)' + p'q' + r$

$\equiv r'p + r'q + p'q' + r$

$\equiv (r + r') \cdot (r + p) + r'q + p'q'$

$\equiv r + p + r'q + p'q'$

$\equiv (r + r')\,(r + q) + (p + p')\,(p + q')$

$\equiv r + q + p + q'$

$\equiv r + p + 1 \equiv 1$

$\therefore$ tautology

6. (b)

The correct translation is

$\forall x[(\text{connected}(x) \wedge \text{bipartite}(x)) \rightarrow \sim \text{planar}(x)]$

however, since $p \rightarrow q \equiv \sim p \vee q$, we can write the above expression also as,

$\forall x[\sim\text{connected}(x) \vee \sim\text{bipartite}(x) \vee \sim \text{planar}(x)]$

7. (a)

If $P \vee Q$ is false, then both P and Q are false.

So, $\neg P \vee \neg Q \equiv \neg F \vee \neg F \equiv T \vee T \equiv T$

$\therefore$ (*i*) is true

Consider (*ii*)

$P \rightarrow (\neg P \rightarrow Q) \equiv P \rightarrow (P^1 \rightarrow Q)$

$\equiv P \rightarrow P + Q$

$\equiv P^1 + P + Q \equiv 1 + Q \equiv 1$

It is a tautology, So (*ii*) is false.

8. (d)

(*i*) Using boolean algebra, we can shown that the given expression reduces to $P + R' + Q'$ which is not a tautology.

(*ii*) For each element $a \in A$, we have n possible choices for value of $f(a)$. Thus there are n^m possible functions.

9. (c)

(*i*) $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$

$\equiv (P' + Q) \rightarrow (Q' + P)$

$\equiv (P' + Q)' + Q' + P$

$\equiv PQ' + Q' + P \equiv P + Q'$

CHAPTER

2

Combinatorics

2.1 INTRODUCTION

Objects (or things) can be arranged in many ways. Suppose there are three objects marked a, b, c on a table from these, two objects can be selected at a line in three different ways as {a, b}, {a, c}, {b, c}. In this way selection of two objects from three objects in three ways is called **Combinations**.

The above selection ab, ac can also be arranged as ab, ba, ac, ca, bc, cb. We can understand that two objects can be selected from three objects and arranged in six ways. These arrangements are called **Permutations**.

Fundamental Concepts

If A is a finite set, then the number of different elements in A is denoted by n(A).e.g.,

If A = {2, 5, 7} then n(A) = 3

If C = ϕ then n(C) = 0

Let us assume that there are three routes say a_1, a_2, a_3 from Delhi to Noida and there are two routes, say b_1, b_2, from Noida to Agra. It may be written as:

A = $\{a_1, a_2, a_3\}$, B = $\{b_1, b_2\}$

n(A) = 3; n(B) = 2

Now we can match the route a_1 from D to N with two routes b_1, b_2 from N, A

i.e., (a_1, b_1), (a_1, b_2)

Similarly the remaining routes can be written as

(a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2).

So to travel from D to A via N, there are 6 different routes

(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_2).

These 6 ways are nothing but the elements of the Cartesian product of the two sets A and B.

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$

$n(A \times B) = 6 = 2 \times 3 = n(A) \times n(B)$.

Fundamental Multiplication Principle of Counting

"Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible out comes and if for each outcome of experiment 1 there are possible out comes of experiment 2, then together there are mn possible outcomes of the two experiments".

The Generalized basic Multiplication Principle of Counting

"If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment and if for each of the possible outcomes the first two experiments there are n_3 possible outcomes of the third experiment and so on, then there is a total of $n_1 \times n_2 \ldots n_r$ possible outcomes of the r experiments".

Keywords to distinguish permutations from combinations:

Permutations : ordered, arrangement, sequence

Combinations : unordered, selection, set

Some useful properties from Number theory used in Combinatorics:

1. **Method for finding the number of positive divisors of a positive integer n:** If a positive integer n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2} \ldots.$ where p_1, p_2 etc. are distinct prime numbers, then the number of positive divisors of n is given by the formula $(n_1 + 1)(n_2 + 1)\ldots$
   For example, the number 80 can be broken as $80 = 2^4 \times 5^1$. So the number of positive divisors of 80 is given by $(4 + 1)(1 + 1) = 10$.
2. **Method for finding the number of numbers from 1 to n, which are relatively prime to n :** The number of numbers from 1 to n, which are relatively prime to n i.e., gcd $(m, n) = 1$, is given by the Euler Totient function $\phi(n)$. If n is broken down into its prime factors as $n = p_1^{n_1} \cdot p_2^{n_2} \ldots.$ where p_1, p_2 etc. are distinct prime numbers, then $\phi(n) = \phi(p_1^{n_1})\,\phi(p_2^{n_2})\ldots$ then by using the property

   $$\phi(p^k) = p^k - p^{k-1}.$$

   we can find each of $\phi(p_1^{n_1}), \phi(p_2^{n_2})\ldots$ etc.

   For *example*, the number of numbers from 1 to n, which are relatively prime to 80 can be found as follows: Since $80 = 2^4 \times 5^1$

The number of numbers from 1 to n, which are relatively prime to 80 = $\phi(80) = \phi(2^4) \times \phi(5^1)$

Now $\phi(2^4) = 2^4 - 2^3 = 16 - 8 = 8$

Similarly, $\phi(5^1) = 5^1 - 5^0 = 5 - 1 = 4$

So, $\phi(80) = 8 \times 4 = 32$

2.2 Permutations

Permutations with no Repetitions

When we select objects from a set consisting of n distinct objects taking each object exactly once (no repetition), and then arrange them in a straight line, this situation is called permutations with no repetition.

The formula for counting this is

$$^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \ldots (n-r+1)$$

Example 2.1 How many 2 letter passwords are there using the letters {*a*, *b*, *c*} if no letter is allowed to be used more then once?

Solution:

$$^3P_2 = \frac{3!}{(3-2)!} = 3 \times 2 = 6$$

The 6 permutations are *ab*, *ba*, *ac*, *ca*, *bc*, *cb*.

Example 2.2 How many ways can four (distinct) dolls be arranged in a straight line?

Solution:

$$^4P_4 = \frac{4!}{(4-4)!} = 4! = 4 \times 3 \times 2 \times 1 = 24 \text{ ways.}$$

Alternately more problems can be solved by box method, which is more general and more powerful.
For example arranging 4 dolls can be thought of as filling 4 boxes corresponding to position of the dolls.
The first box can be filled in 4 ways. Since repetition is not allowed, the second box can be filled only in 3 ways.
The third box in 2 ways and last box in only 1 way.
$\therefore$ Total arrangements = $4 \times 3 \times 2 \times 1$ = 24 ways.

Permutations with Unlimited Repetition

When we select an object, from a set of distinct objects, taking each object any number of times (unlimited repetition) and arrange them in a straight line, this situation is called permutation with unlimited repetition. The formula for counting this is n^r.

Example 2.3 How many 2 letter passwords can be made from {*a*, *b*, *c*}, if a letter can be used any number of times?

Solution:

$$3^2 = 9 \text{ passwords}$$

The nine permutations are : *aa*, *ab*, *ac*, *ba*, *bb*, *bc*, *ca*, *cb*, *cc*.

NOTE

For objects such as passwords & number, if nothing is mentioned regarding repetition, the default assumption is that unlimited repetition is allowed.
Using box method we make the password by filling 2 boxes. Each can be filled in 3 ways (since repetition allowed) $3 \times 3 = 9$ passwords.

Example 2.4 If there are 10 multiple choice question with four choices for each question, How many answer sheets are possible?

Solution:

5^{10} answer sheets. We can think of each of the question as a box and each of the 10 boxes can be filled is 5 ways (Choice *a*, *b*, *c* or *d* or leave it blank):
$\therefore$ The number of ways of filling up all 10 boxes is $5 \times 5 \times 5 \times$ ... 10 times = 5^{10}
The box method is very powerful for use in all permutation problems (with repetition or without repetition).

Student's Assignment

Q.1 How many strings are there of lowercase letters of length four or less?

Q.2 A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

(a) How many socks must he take out to be sure that he has at least two socks of the same color?

(b) How many socks must he take out to be sure that he has at least two black socks?

Q.3 A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

(a) How many balls must she select to be sure of having at least three balls of the same color?

(b) How many balls must she select to be sure of having at least three blue balls?

Q.4 What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Q.5 Atleast how many numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add up to 7?

Q.6 How many subsets with an odd number of elements does a set with 10 elements have?

Q.7 A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

(a) are there in total?

(b) contain exactly three heads?

(c) contain at least three heads?

(d) contain the same number of heads and tails?

Q.8 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Q.9 Find the coefficient of x^5y^8 in $(x + y)^{13}$.

Q.10 How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?

Q.11 What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 2y)^{200}$?

Q.12 In how many ways can 5 numbers be selected and arranged in ascending order from the set {1,2,3...10}?

Q.13 In how many different ways can 5 ones and 20 twos be permuted so that each one is followed by atleast 2 twos?

Q.14 How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

Q.15 A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose

(a) six bagels?

(b) a dozen bagels?

(c) a dozen bagels with at least one of each kind?

(d) a dozen bagels without least three egg bagels and no more than two salty bagels?

Q.16 How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

Q.17 How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = 1, 2, 3, 4, 5$, is a non-negative integer such that

(a) $x_1 \geq 1$

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$

(c) $0 \leq x_1 \leq 10$

Q.18 How many different bit strings can be transmitted if the string must begin with a 1 bit, must include three additional 1 bits (so that a total of four 1 bits is sent), must include a total of twelve 0 bits, and must have at least two 0 bits following each 1 bit?

Q.19 An agency has 10 available foster families $F_1...F_{20}$ and 6 children $C_1...C_6$ to place. In how many ways can they do this if
(a) No family can get more than one child.
(b) A family can get more than one child.

Q.20 How many distinguishable permutation can be generated from word "BANANA"?
(a) 720 (b) 60
(c) 240 (d) 120

Q.21 How many integers in S = {1, 2, 3, ..., 1000} are divisible by 3 or 5?
(a) 599 (b) 467
(c) 333 (d) 66

Q.22 How many numbers from a set {1, 2, 3, ..., 20} should be chosen in order to be sure to have one number multiple of another.
(a) 10 (b) 11
(c) 3 (d) 9

Q.23 How many numbers must be chosen from set {1, 2, 3, ..., 8}, such that atleast 2 of them must have Sum = 9
(a) 28 (b) 9
(c) 5 (d) 10

Q.24 In a string of length n, with distinct letters, how many substrings can be generated (other than null string)?
(a) 2^n (b) n^2
(c) $\frac{n(n+1)}{2}$ (d) $\frac{n(n-1)}{2}$

Answer Key:

20. (b) **21.** (b) **22.** (b) **23.** (c) **24.** (c)

Student's Assignments — Explanations

1. (Sol).

Here repetition is allowed by default (nothing is mentioned about repetition).
Number of possible strings of 0 length (empty string) : 1
Number of possible strings of length 1 : 26
Number of possible strings of length 2 : 26 × 26
Number of possible strings of length 3 : 26 × 26 × 26
Number of possible strings of length 3 : 26 × 26 × 26 × 26
Therefore the total number of strings of length 4 or less = $1+26 + 26^2 + 26^3 + 26^4$

2. (Sol).

(a) He must take out three socks to make sure that he gets a pair. Because the first socks could be of one colour and second socks could be of another colour.
Hence the third socks that he draws from the drawer is of one of the colour of the two socks that he had drawn earlier, which is sufficient to make a pair.

(b) He must take out 14 socks to be sure that he has at least 2 black socks. Because there are 12 black and 12 brown socks in the drawer. So when he draws first 12 socks they could be all brown and hence 2 more socks need to be drawn to make sure a pair of black socks.

3. (Sol).

(a) She must select five balls to be sure of having at least 3 balls of the same colour. This is because first 4 attempts have the following possibilities which does not ensure three balls of same colour.
Red, Blue, Red, Blue or
Blue, Red, Red, Blue ...and so on
Hence we need fifth ball to ensure three balls of same colour.

(b) Thirteen balls must be selected to be sure of having atleast three blue balls.
The first ten balls that she draws could be all ten red balls. Hence she need to draw three more balls to ensure three blue balls.

4. (Sol).

Here the number of students are similar to number of pigeons and number of states are similar to pigeon holes.

Number of pigeons = n
Number of pigeons hole = 50

Therefore $\left\lfloor \frac{n-1}{50} \right\rfloor + 1 = 100$

The minimum value of n is obtained after removing the floor function and solving for n.
$\Rightarrow$ n = 4951 pigeons (students).

5. (Sol).

The number of pairs that add upto 7 are the pigeon holes.
{1, 6}, {2, 5}, {3, 4} are the holes.
Let 'N' be the number of such numbers. If two numbers sit in any one of these pigeon holes, we have a pair whose total is 7.

Therefore $\left\lfloor \frac{N-1}{3} \right\rfloor + 1 = 2$

$\Rightarrow$ Atleast N = 4 numbers must be selected.

6. (Sol).

The number of subsets with 1 element = ${}^{10}C_1$
The number of subsets with 3 elements = ${}^{10}C_3$
The number of subsets with 5 elements = ${}^{10}C_5$
The number of subsets with 7 elements = ${}^{10}C_7$
The number of subsets with 9 elements = ${}^{10}C_9$
Hence the total number of subsets with an odd number of elements are:

$${}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9 = 2^{10-1} = 2^9 = 512$$

7. (Sol).

When a coin is tossed there are only 2 possibilities (H and T).

(a) Total number of possibilities are:
$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8$

(b) The total number of possibilities with exactly three heads are: ${}^8C_3$ (any of the 3 coins can turn up heads).

(c) The total number of possibilities with atleast three heads are: $2^8 - \{{}^8C_0 + {}^8C_1 + {}^8C_2\}$.

(d) The total number of possibilities with same number of heads and tails are: ${}^8C_4$ (any of the 4 coins could be head and remaining 4 coins could be tails).

8. (Sol).

First, note that this expression equals $(2x + (-3y))^{25}$. By the Binomial Theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

Consequently, the coefficient of $x^{12} y^{13}$ in the expression is obtained when j = 13, namely,

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25}{13!\,12!} 2^{12} 3^{13}$$

9. (Sol).

The coefficient of $x^{n_1} y^{n_2}$ is $\frac{n}{n_1!\,n_2!}$. Therefore the coefficient is $\frac{13!}{5!\,8!}$ which is = $13 \times 11 \times 9$ = 1287.

10. (Sol).

We will get like terms when we expand $(x+y)^{100}$ as $(x + y)(x + y)(x + y)(x + y)$ ...100 times.
The binomial expansion collects all like terms.
In the expansion the terms are in the form $x^{n_1} y^{n_2}$ with the condition that $n_1 + n_2 = 100$ always. Thus, the number of solution of this equation gives the number of terms in the binomial expansion.

$${}^{2-1+100}C_{100} = 101$$

$\therefore$ The number of solutions of this equation gives the number of terms are 101.

11. (Sol).

$$\frac{200!}{100!\,99!}(2x)^{101}(-2y)^{99}$$

$$= -\frac{200!}{100!\,99!} \times 2^{200} \times x^{101} \times y^{99}$$

12. (Sol).

Since, after selection there is only one way to put 5 numbers in ascending order, the answer is ${}^{10}C_5 \times 1 = {}^{10}C_5 = 252$.