CIVIL ENGINEERING
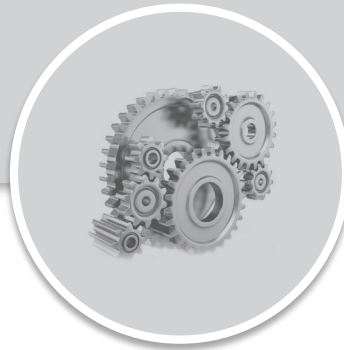
Engineering Mechanics

Comprehensive Theory
with Solved Examples *and* Practice Questions

MADE EASY
Publications

Engineering Mechanics
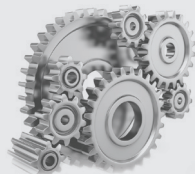
CONTENTS
Engineering Mechanics

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Basics of Vectors

1.1 VECTORS AND SCALARS

Many physical quantities are completely described by a numerical value alone and are added according to the ordinary rules of algebra. As an example the mass of a system is described by saying that it is 5 kg. If two bodies one having a mass of 5 kg and the other having a mass of 2 kg are added together to make a composite system, the total mass of the system becomes 5 kg + 2 kg = 7 kg. Such quantities are called scalars.

So, a scalar is any positive or negative physical quantity that can be completely specified by its magnitude. Other examples of scalar quantities are length, mass and time.

While the complete description of certain physical quantities requires a numerical value as well as a direction in space. Velocity of a particle is an example of this kind. The magnitude of velocity is represented by a number such as 5 m/s and tells us how fast a particle is moving. But the description of velocity becomes complete only when the direction of velocity is also specified. We can represent this velocity by drawing a line parallel to the velocity and putting an arrow showing the direction of velocity.
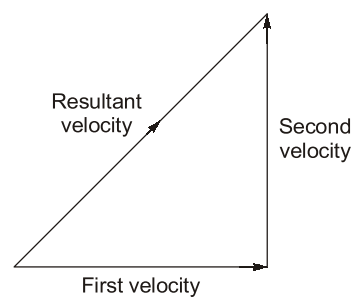
Fig. Triangle law to obtain resultant velocity

Further, if a particle is given two velocities simultaneously, its resultant velocity is different from the two velocities and is obtained by using a special rule known as triangle law.

The physical quantities which have magnitude and direction and which can be added according to the laws of vector addition are called vector quantities. Other examples of vector quantities are force, linear momentum, electric field, magnetic field etc.

1.2 EQUALITY OF VECTORS

Two vectors (representing two values of the same physical quantity) are called equal if their magnitudes and directions are same. Thus, a parallel translation of a vector does not bring about any change in it.

1.3 UNIT VECTORS

A vector whose magnitude is unity (i.e. 1 unit) is called a unit vector. Unit vector is a director vector. Unit vector in the direction of any given vector $\vec{a}$ is given as $\dfrac{\vec{a}}{|a|}$.

EXAMPLE : 1.1 The unit vector in the direction of $\vec{A} = 5\hat{i} + \hat{j} - 2\hat{k}$ is

Solution:

$$\left|\vec{A}\right| = \sqrt{5^2 + 1^2 + (-2)^2} = \sqrt{30}$$

The required unit vector is $\dfrac{\vec{A}}{\left|\vec{A}\right|} = \dfrac{5}{\sqrt{30}}\hat{i} + \dfrac{1}{\sqrt{30}}\hat{j} - \dfrac{2}{\sqrt{30}}\hat{k}$

1.4 ADDITION OF VECTORS

The triangle rule of vector addition is already described above. If $\vec{a}$ and $\vec{b}$ are two vectors to be added, a diagram is drawn in which the tail of $\vec{b}$ coincides with the head of $\vec{a}$. The vector joining the tail of $\vec{a}$ with the head of $\vec{b}$ is the vector sum of $\vec{a}$ and $\vec{b}$.

The same rule may be stated in a slightly different way by parallelogram law.
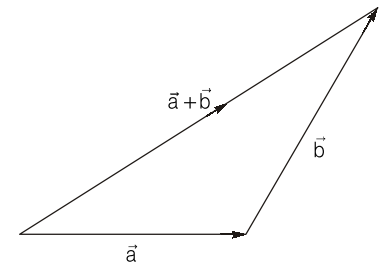
Fig. Vector addition as per triangle law

In parallelogram law we draw vectors $\vec{a}$ and $\vec{b}$ with both the tails coinciding as shown in figure. Taking these two as the adjacent sides we complete the parallelogram. The diagonal through the common tails gives the sum of the two vectors.

Suppose the magnitude of $\vec{a} = a$ and that of $\vec{b} = b$. If the angle between $\vec{a}$ and $\vec{b}$ is θ, then

$$AD^2 = (AB + BE)^2 + (DE)^2$$
$$= (a + b\cos\theta)^2 + (b\sin\theta)^2$$
$$= a^2 + 2ab\cos\theta + b^2$$

Thus, the magnitude of $\vec{a} + \vec{b}$ is

$$= \sqrt{a^2 + b^2 + 2ab\cos\theta}$$

Its angle with $\vec{a}$ is α where, $\tan\alpha = \dfrac{DE}{AE} = \dfrac{b\sin\theta}{a + b\cos\theta}$
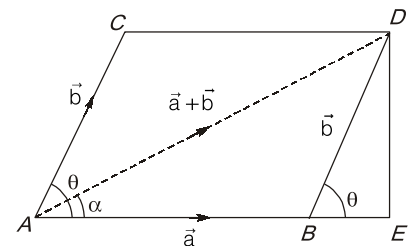
Fig. Vector addition as per parallelogram law

Special cases:

(a) When two vectors are acting in same direction, then, $\theta = 0°$

∴ $$\left|\vec{a} + \vec{b}\right| = \sqrt{a^2 + b^2 + 2ab} = a + b$$

and
$$\tan\alpha = \frac{b \times \sin 0°}{a + b\cos 0°} = 0$$
$$\Rightarrow \qquad \alpha = 0°$$

Thus, the magnitude of sum of vectors $\vec{a}$ and $\vec{b}$ is equal to the sum of magnitudes of two vectors acting in same direction and their resultant acts in direction of $\vec{a}$ and $\vec{b}$.

(b) When two vectors acts in opposite directions:

Then,
$$\theta = 180°$$
$$\therefore \qquad \left|\vec{a} + \vec{b}\right| = \sqrt{a^2 + b^2 - 2ab} = a - b$$

and
$$\tan\alpha = \frac{b \times \sin(180°)}{a + b \times \cos(180°)} = 0$$
$$\Rightarrow \qquad \alpha = 0° \text{ or } 180°$$

Thus, the magnitude of sum of the vectors $\vec{a}$ and $(-\vec{b})$ is equal to the difference of magnitudes of two vectors and their resultant acts in direction of bigger vector.

1.5 MULTIPLICATION OF VECTOR BY A NUMBER

Suppose $\vec{a}$ is a vector of magnitude a and k is a number. We define the vector $\vec{b} = k\vec{a}$ as a vector of magnitude $|ka|$. If k is positive the direction of the vector $\vec{b} = k\vec{a}$ is same as that of $\vec{a}$. If k is negative, the direction of $\vec{b}$ is opposite to $\vec{a}$. In particular, multiplication by (–1) just inverts the direction of the vector. The vectors $\vec{a}$ and $-\vec{a}$ have equal magnitudes but opposite directions.

If $\vec{a}$ is a vector of magnitude a and $\hat{u}$ is a vector of unit magnitude in the direction of $\vec{a}$, we can write $\vec{a} = a\hat{u}$.

1.6 SUBTRACTION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two vectors. We define $\vec{a} - \vec{b}$ as the sum of the vector $\vec{a}$ and the vector $(-\vec{b})$. To subtract $\vec{b}$ from $\vec{a}$, invert the direction of $\vec{b}$ and add to $\vec{a}$.

Fig. Vector diagram representing $\vec{a} + (-\vec{b})$

1.7 RESOLUTION OF VECTORS

Consider a vector $\vec{a} = \overrightarrow{OA}$ in the X-Y plane drawn from the origin O. The length OB is called the projection of $\overrightarrow{OA}$ on X-axis. Similarly OC is the projection of $\overrightarrow{OA}$ on Y-axis. According to the rules of vector addition

$$\vec{a} = \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$$

We have resolved the vector $\vec{a}$ into two parts, one along OX and the other along OY. The magnitude of the part along OX is OB = $a\cos\alpha$ and the magnitude of the part along O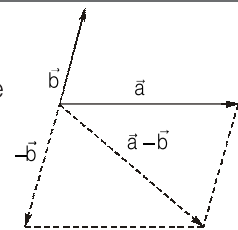Y is OC = $a\cos\beta$. If $\vec{i}$ and $\vec{j}$ denote vectors of unit magnitude along OX and OY respectively, we get

$$\overrightarrow{OA} = a\cos\alpha\,\hat{i} \text{ and } \overrightarrow{OC} = a\cos\beta\,\hat{j}$$

So, that,
$$\vec{a} = a\cos\alpha\,\hat{i} + a\cos\beta\hat{j}$$

Fig. Resolution of vector $\overrightarrow{OA}$ in x and y directions

If the vector $\vec{a}$ is not in the X-Y plane, it may have nonzero projections along X, Y, Z axes and we can resolve it into three parts i.e. along the X, Y and Z axes. If α, β, γ be the angles made by the vector $\vec{a}$ with the three axes respectively, we get

$$\vec{a} = a\cos\alpha\,\hat{i} + a\cos\beta\hat{j} + a\cos\gamma\hat{k}$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along X, Y and Z axes respectively. The component of vector $\vec{a}$ along direction making angle θ with it is $a\cos\theta$ which is the projection of $\vec{a}$ along the given direction.

Fig. Horizontal projection of $\vec{a}$

EXAMPLE : 1.2

A force of 10.5 N acts on a particle along a direction making an angle of 37° with the vertical. Find the component of the force in the vertical direction?

Solution:
The component of the force in the vertical direction will be

$$F_\perp = F\cos\theta = (10.5\,\text{N})\,(\cos37°)$$

$$F_\perp = 10.5 \times \frac{4}{5} = 8.40\,\text{N}$$

EXAMPLE : 1.3

The magnitudes of vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ in the figure shown are equal. Find the direction of $\overrightarrow{OA} + \overrightarrow{OB} - \overrightarrow{OC}$.

Solution:
Let,
$$OA = OB = OC = F$$

$$x\text{-component of } \overrightarrow{OA} = F\cos30° = F\frac{\sqrt{3}}{2}$$

$$x\text{-component of } \overrightarrow{OB} = F\cos60° = \frac{F}{2}$$

$$x\text{-component of } \overrightarrow{OC} = F\cos135° = -\frac{F}{\sqrt{2}}$$
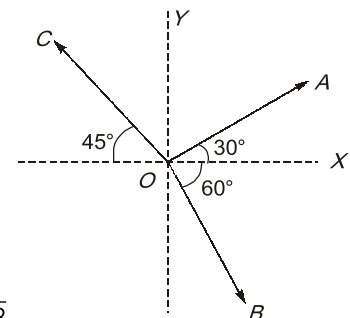
$$x\text{-component of } \overrightarrow{OA} + \overrightarrow{OB} - \overrightarrow{OC} = \left(\frac{F\sqrt{3}}{2}\right) + \left(\frac{F}{2}\right) - \left(-\frac{F}{\sqrt{2}}\right) = \frac{F}{2}\left(\sqrt{3} + 1 + \sqrt{2}\right)$$

$$y\text{-component of } \overrightarrow{OA} = F\cos 60° = \frac{F}{2}$$

$$y\text{-component of } \overrightarrow{OB} = F\cos 120° = -\frac{F\sqrt{3}}{2}$$

$$y\text{-component of } \overrightarrow{OC} = F\cos 45° = \frac{F}{\sqrt{2}}$$

$$y\text{-component of } \overrightarrow{OA} + \overrightarrow{OB} - \overrightarrow{OC} = \left(\frac{F}{2}\right) + \left(-\frac{F\sqrt{3}}{2}\right) - \left(\frac{F}{\sqrt{2}}\right) = \frac{F}{2}\left(1 - \sqrt{3} - \sqrt{2}\right)$$

Angle of $\overrightarrow{OA} + \overrightarrow{OB} - \overrightarrow{OC}$ with the x-axis

$$= \tan^{-1}\frac{\frac{F}{2}\left(1 - \sqrt{3} - \sqrt{2}\right)}{\frac{F}{2}\left(1 + \sqrt{3} + \sqrt{2}\right)} = \tan^{-1}\frac{\left(1 - \sqrt{3} - \sqrt{2}\right)}{\left(1 + \sqrt{3} + \sqrt{2}\right)}$$

| **EXAMPLE : 1.4** | If the radius of the circle shown in the figure is R, then the resultant of the three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ is |

(a) $R\left(1 + \sqrt{2}\right)$ (b) $2R\left(1 + \sqrt{2}\right)$

(c) $3R\left(1 + \sqrt{2}\right)$ (d) $R\left(1 + 2\sqrt{2}\right)$
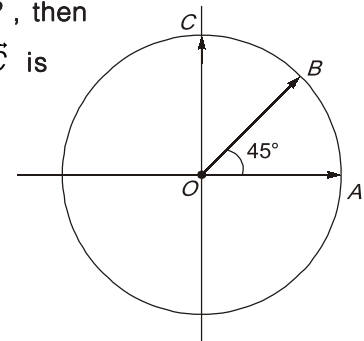
Solution: (a)

$$OA = OC$$

$\overrightarrow{OA} + \overrightarrow{OC}$ is along $\overrightarrow{OB}$ (bisector) and its magnitude is

$$2R\cos 45° = R\sqrt{2}$$

$\left(\overrightarrow{OA} + \overrightarrow{OC}\right) + \overrightarrow{OB}$ is along $\overrightarrow{OB}$ and its magnitude is

$$R\sqrt{2} + R = R\left(1 + \sqrt{2}\right)$$

1.8   DOT PRODUCT OR SCALAR PRODUCT OF TWO VECTORS

The dot product (also called scalar product) of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \cdot \vec{b} = ab\cos\theta$$

The dot product is commutative and distributive.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$
$$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

Fig. Vector $\vec{a}$ and $\vec{b}$ having angle θ between them

OBJECTIVE BRAIN TEASERS

Q.1 A particle whose speed is 25 m/sec moves along the line from $A(2, 1)$ to $B(9, 25)$. The velocity vector of the particle in the form $a\hat{i} + b\hat{j}$ is
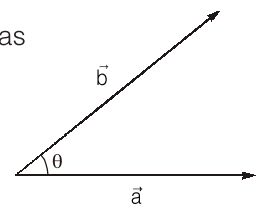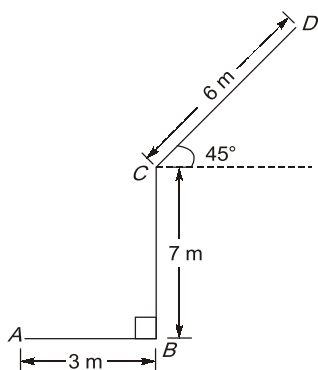
(a) $\left(7\hat{i} + 7\hat{j}\right)$ m/s   (b) $\left(7\hat{i} + 24\hat{j}\right)$ m/s

(c) $\left(24\hat{i} + 12\hat{j}\right)$ m/s   (d) $\left(24\hat{i} + 7\hat{j}\right)$ m/s

Q.2 A particle moves along a path ABCD as shown in the figure. The magnitude of net displacement of the particle from A to D is

(a) 9.26 m   (b) 13.37 m
(c) 10.42 m   (d) 8.38 m

Q.3 The angle between two vectors $\vec{A} = 2\hat{i} + \hat{j} - \hat{k}$

and $\vec{B} = \hat{i} - \hat{k}$ is

(a) 30°   (b) 60°
(c) 0°   (d) 90°

Q.4 If two vectors are given as:

$\vec{A} = \hat{i} + 2\hat{j} - \hat{k}$

$\vec{B} = -\hat{i} - \hat{j} + \hat{k}$

then, the vector perpendicular to $\left(\vec{A} \times \vec{B}\right)$ can be

(a) $2\hat{i} + 4\hat{j} + 2\hat{k}$ (b) $\hat{i} + \hat{j} + \hat{k}$

(c) $25\hat{i} - 625\hat{j} - 25\hat{k}$ (d) $3\hat{i} - 2\hat{j} + 3\hat{k}$

Q.5 If $\left(\vec{a} + \vec{b}\right)$ is perpendicular to $\vec{b}$ and $\vec{a} + 2\vec{b}$ is

perpendicular to $\vec{a}$. If $\left|\vec{a}\right| = a$ and $\left|\vec{b}\right| = b$, then

(a) $a = b$ (b) $a = 2b$
(c) $b = 2a$ (d) $a = b\sqrt{2}$

Q.6 Two forces are acting on a body, $\vec{F_1} = 2\hat{i} + 3\hat{j}$

and it does 8J of work, $\vec{F_2} = 3\hat{i} + 5\hat{j}$ and it does

−4J of work on body. The magnitude of displacement traversed by the body is
(a) 57.27 m   (b) 30.53 m
(c) 54.40 m   (d) 61.06 m

ANSWER KEY

1. (b)   **2.** (b)   **3.** (a)   **4.** (c)   **5.** (d)

6. (d)

HINTS & EXPLANATIONS

1.   **(b)**
Velocity vector is given by the product of magnitude of velocity (speed) multiplied by the unit vector along velocity.
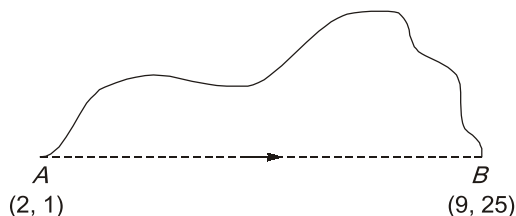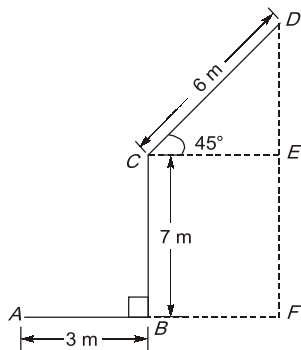

Unit vector along velocity,

$$\widehat{AB} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{(9-2)\hat{i} + (25-1)\hat{j}}{\sqrt{7^2 + 24^2}}$$

$$\vec{V} = \left|\vec{V}\right|\widehat{AB} = 25\left(\frac{7\hat{i} + 24\hat{j}}{25}\right)$$

$$= \left(7\hat{i} + 24\hat{j}\right) \text{m/s}$$

2.   (b)

The displacement of the particle from A to D is given by AD

$$AD = \sqrt{AF^2 + DF^2} = \sqrt{(AB+BF)^2 + (DE+EF)^2}$$

$$= \sqrt{(3+6\cos45°)^2 + (7+6\sin45°)^2}$$

$$= \sqrt{(7.24)^2 + (11.24)^2} = \sqrt{52.41 + 126.33}$$

$$= \sqrt{178.74} = 13.37 \text{ m}$$

Alternatively (vector method):
Let, $A(0, 0)$
$D(3 + 6\cos45, 7 + 6\sin45)$

$$\vec{r} = \overrightarrow{DA}$$

$$= (3+3\sqrt{2})\hat{i} + (7+3\sqrt{2})\hat{j}$$

$$\left|\vec{r}\right| = 13.356 \text{ m}$$

3.   (a)

The dot product of two vectors is given by

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

$$\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}\right]$$

$$\theta = \left[\frac{(2\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}-\hat{k})}{\sqrt{2^2+1^2+(-1)^2}\sqrt{1^2+(-1)^2}}\right]$$

$$= \cos^{-1}\left[\frac{2+1}{\sqrt{6}\sqrt{2}}\right] = \cos^{-1}\left[\frac{3}{2\sqrt{3}}\right]$$

$$= \cos^{-1}\left[\frac{\sqrt{3}}{2}\right] = 30°$$

4.   (c)

The cross product of $\left(\vec{A}\times\vec{B}\right)$ is given as

$$\left(\vec{A}\times\vec{B}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}(1) - \hat{j}(0) + \hat{k}(-1+2) = \hat{i} + \hat{k}$$

$\vec{C}$ will be perpendicular to $\left(\vec{A}\times\vec{B}\right)$, it $\vec{C}\bullet\left(\vec{A}\times\vec{B}\right)$ gives a result zero. From options,

$$\left(25\hat{i} - 625\hat{j} - 25\hat{k}\right)\left(\hat{i}+\hat{k}\right)$$

$$\therefore \quad 25 - 25 = 0$$

5.   (d)

From the given conditions, $\left(\vec{a}+\vec{b}\right)$ is perpendicular to $\vec{b}$, therefore their dot product is zero.

$$\left(\vec{a}+\vec{b}\right)\cdot\vec{b} = 0$$

$$\vec{a} \cdot \vec{b} + \left|\vec{b}\right|^2 = 0$$

$$\vec{a} \cdot \vec{b} + b^2 = 0 \qquad \dots \text{(i)}$$

Similarly,

$$\left(\vec{a} + 2\vec{b}\right) \cdot \vec{a} = 0$$

$$\left|\vec{a}\right|^2 + 2\vec{a} \cdot \vec{b} = 0$$

$$a^2 + 2\vec{a} \cdot \vec{b} = 0 \qquad \dots \text{(ii)}$$

From equations (i) and (ii), we get

$$-\frac{a^2}{2} = -b^2$$

$$\therefore \qquad a = \pm\sqrt{2}b$$

Neglecting negative sign as magnitude cannot be negative, therefore $a = \sqrt{2}b$.

6.   (d)

Let the displacement of the body is given by

$$\vec{r} = x\hat{i} + y\hat{j}$$

work done by first force,

$$W_1 = \vec{F_1} \cdot \vec{r} = 8$$

$$\therefore \left(2\hat{i} + 3\hat{j}\right) \cdot \left(x\hat{i} + y\hat{j}\right) = 8$$

$$2x + 3y = 8 \qquad \dots \text{(i)}$$

Work done by second force,

$$W_2 = \vec{F_2} \cdot \vec{r} = -4$$

$$\left(3\hat{i} + 5\hat{j}\right) \cdot \left(x\hat{i} + y\hat{j}\right) = -4$$

$$3x + 5y = -4 \qquad \dots \text{(ii)}$$

Solving (i) and (ii) equation, we get

$$x = 52, \; y = -32$$

$$\therefore \qquad \vec{r} = 52\hat{i} - 32\hat{j}$$

$$\left|\vec{r}\right| = \sqrt{(52)^2 + (-32)^2} = \sqrt{3728}$$

$$= 61.06 \, \text{m}$$

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