UPPSC-AE 2021

Uttar Pradesh
Public Service Commission

Combined State Engineering Services Examination

Assistant Engineer

Electrical Engineering

Electromagnetic Theory

Well Illustrated **Theory** *with*
Solved Examples and **Practice Questions**

MADE EASY Publications

Electromagnetic Theory

Contents

CHAPTER 1

Vector Analysis

1.1 Introduction

The quantities of interest appearing in the study of EM theory can almost be classified as either a scalar or a vector. Quantities that can be described by a magnitude alone are called scalars. Distance, temperature, mass etc. are examples of scalar quantities. Quantities, that require both a magnitude and a direction to fully characterize them are vectors. Vector quantities include velocity, force, acceleration etc are examples of vector quantities.

In electromagnetics, we frequently use the concept of a **field**. A field is a function that assigns a particular physical quantity to every point in a region. In general, a field varies with both position and time. There are scalar fields and vector fields. Temperature distribution in a room and electric potential are examples of scalar fields. Electric field and magnetic flux density are examples of vector fields.

NOTE: Vectors are denoted by an arrow over a letter ($\vec{A}$) and scalars are denoted by simple letter (A).

1.1.1 Unit Vector

- A unit vector $\hat{a}_A$ along $\vec{A}$ is defined as a vector whose magnitude is unity (*i.e.*, 1) and its direction is along $\vec{A}$, that is

$$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$

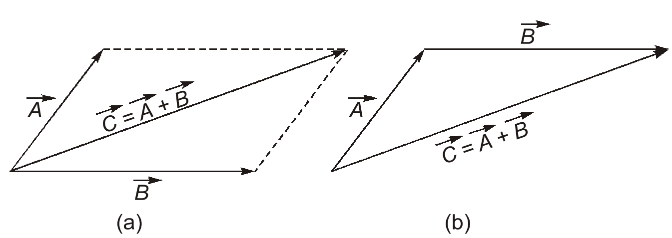
Thus we can write $\vec{A}$ as $\quad \vec{A} = A\,\hat{a}_A = |\vec{A}|\hat{a}_A$

REMEMBER: Any vector can be written as product of its magnitude and its unit vector.

1.1.2 Vector Addition and Subtraction

Two vectors $\vec{A}$ and $\vec{B}$ can be added together to give another vector $\vec{C}$; that is,

$$\vec{C} = \vec{A} + \vec{B}$$



Vector addition (a) parallelogram rule, (b) head-to-tail rule.

- Vector subtraction is similarly carried out as

$$\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

NOTE

- $\vec{A}+\vec{B}=\vec{B}+\vec{A}$ (Commutative law)
- $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$ (Associative law)
- $k(\vec{A}+\vec{B})=k\vec{A}+k\vec{B}$ (Distributive law)
- $\dfrac{\vec{A}+\vec{B}}{k}=\dfrac{1}{k}\vec{A}+\dfrac{1}{k}\vec{B}$

1.1.3 Position and Distance Vectors:

- A point P in Cartesian coordinates may be represented by (x, y, z).
- The position vector $\vec{r}_p$ (or radius vector) of point P is defined as the directed distance from origin O to P.

$$\vec{r}_p = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

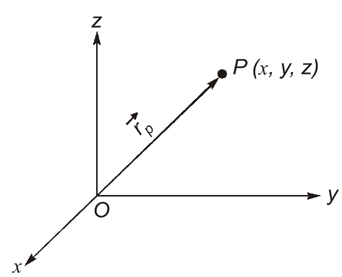
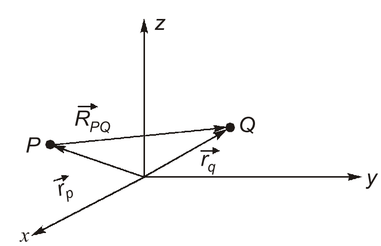


Illustration of position vector $\vec{r}_p = x\hat{a}_x + \hat{y}a_y + z\hat{a}_z$



Vector distance $\vec{R}_{PQ}$

- The distance vector is the displacement from one point to another.
- Consider point P with position vector $\vec{r}_p$ and point Q with position vector $\vec{r}_q$. The displacement from P to Q is written as

$$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p$$

Example - 1.1 Point P and Q are located at (0, 2, 4) and (–3, 1, 5). The position vector P will be

(a) $-3\hat{a}_x + \hat{a}_y + 5\hat{a}_z$ (b) $-3\hat{a}_x + 5\hat{a}_z$
(c) $2\hat{a}_y + 4\hat{a}_z$ (d) $2\hat{a}_x - 4\hat{a}_z$

Solution: (c)

$$\vec{r}_p = 0\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z = 2\hat{a}_y + 4\hat{a}_z$$

Example - 1.2 Point P and Q are located at (0, 2, 4) and (–3, 1, 5). The distance vector from P to Q will be

(a) $-3\hat{a}_x - \hat{a}_y + \hat{a}_z$ (b) $-3\hat{a}_x - \hat{a}_y - \hat{a}_z$
(c) $3\hat{a}_x + \hat{a}_y + \hat{a}_z$ (d) $3\hat{a}_x - \hat{a}_y + \hat{a}_z$

Solution: (a)

$$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p = (-3, 1, 5) - (0, 2, 4) = (-3, -1, 1)$$
$$= -3\hat{a}_x - \hat{a}_y + \hat{a}_z$$

1.1.4 Vector Multiplication

- When two vectors are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication.
  1. Scalar (or dot) product : $\vec{A} \cdot \vec{B}$
  2. Vector (or cross) product : $\vec{A} \times \vec{B}$

     Multiplication or three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ can result in either
  3. Scalar triple product : $\vec{A} \cdot (\vec{B} \times \vec{C})$
  4. Vector triple product : $\vec{A} \times (\vec{B} \times \vec{C})$

Dot Product:

- The dot product, or the scalar product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} \cdot \vec{B}$ is defined geometrically as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle between them.

$$\vec{A} \cdot \vec{B} = AB\cos\theta_{AB}$$

Where θ_{AB} is the smaller angle between $\vec{A}$ and $\vec{B}$. The result of $\vec{A} \cdot \vec{B}$ is called either the scalar product because it is scalar, or the dot product due to the dot sign.

If $\vec{A} = (A_x, A_y, A_z)$

and $\vec{B} = (B_x, B_y, B_z)$

then $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$

NOTE: Two vectors $\vec{A}$ and $\vec{B}$ are said to be orthogonal (or perpendicular) with each other if $\vec{A} \cdot \vec{B} = 0$.

- The dot product obeys the following laws:

  Commutative Law

  Expression: $\vec{A} \times \vec{B} = \vec{B} \times \vec{A}$

  Distributive Law

  Expression: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

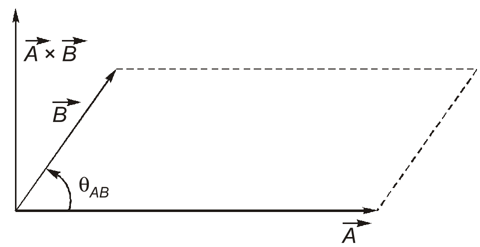
  $$\vec{A} \cdot \vec{A} = |\vec{A}|^2 = |A|^2$$

NOTE

- $\hat{a}_x \cdot \hat{a}_y = \hat{a}_y \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_x = 0$
- $\hat{a}_x \cdot \hat{a}_x = \hat{a}_y \cdot \hat{a}_y = \hat{a}_z \cdot \hat{a}_z = 1$

Cross Product:

- The cross product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} \times \vec{B}$, is a vector quantity whose magnitude is the area of the parallelepiped formed by $\vec{A}$ and $\vec{B}$ and is in the direction of advance of the right-handed screw as $\vec{A}$ is turned into $\vec{B}$.



The cross product of $\vec{A}$ and $\vec{B}$ is a vector with magnitude equal to the area of parallelogram and the direction as indicated

$$\vec{A} \times \vec{B} = AB \sin \theta_{AB} \, \hat{a}_n$$

where $\hat{a}_n$ is a unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$.

- The vector multiplication of equation is called **cross product** due to the cross sign. It is also called **vector product** because the result is a vector.

  If $\vec{A} = (A_x, A_y, A_z)$ and $B = (B_x, B_y, B_z)$ then :

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{a}_x + (A_z B_x - A_x B_z)\hat{a}_y + (A_x B_y - A_y B_x)\hat{a}_z$$

  Also $\vec{A} \times \vec{B} = 0$, then $\sin \theta_{AB} = 0°$ or $180°$; this shows that $\vec{A}$ and $\vec{B}$ are parallel or antiparallel to each other.

- Above result is obtained by 'crossing' terms in cyclic permutation, hence the name cross product. Note that the cross product has the following properties

  1. It is not commutative:

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

  2. It is not associative:

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

  3. It is distributive:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

NOTE

For a vector $A = \hat{a}_x + \hat{a}_y + \hat{a}_z$

- $\vec{A} \times \vec{A} = 0$
- $\hat{a}_x \times \hat{a}_y = \hat{a}_z$, $\hat{a}_y \times \hat{a}_z = \hat{a}_x$, $\hat{a}_z \times \hat{a}_x = \hat{a}_y$

Scalar Triple Product:

- Given three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we define scalar triple product as,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

- If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ and $\vec{C} = (C_x, C_y, C_z)$, then $\vec{A} \cdot (\vec{B} \times \vec{C})$ is the volume of a parallelepiped having $\vec{A}$, $\vec{B}$, and $\vec{C}$ as edges and is easily obtained by finding the determinant of the 3×3 matrix formed by $\vec{A}$, $\vec{B}$, and $\vec{C}$; that is

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

- Since the result of this vector multiplication is scalar these two equations are called the scalar triple product.

Vector Triple Product:

- For vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we define the vector triple product as

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\,.\,\vec{C}) - \vec{C}(\vec{A}.\vec{B})$$

This is obtained using the "bac-cab" rule.

Example - 1.3 Three field quantities are given by $\vec{P} = 2\hat{a}_x - \hat{a}_z$ and $\vec{Q} = 2\hat{a}_x - \hat{a}_y + 2\hat{a}_z$, $\vec{R} = 2\hat{a}_x - 3\hat{a}_y + \hat{a}_z$. The value of $(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q})$ is

(a) $2\hat{a}_x - 12\hat{a}_y + 4\hat{a}_z$ (b) $2\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z$

(c) $2\hat{a}_x - 12\hat{a}_y - 4\hat{a}_z$ (d) $-2\hat{a}_x - 12\hat{a}_y - 4\hat{a}_z$

Solution: (b)

$$(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q}) = 2(\vec{Q}\times\vec{P})$$

$$= 2\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= 2(1-0)\,\hat{a}_x + 2\,(4+2)\,\hat{a}_y + 2(0+2)\,\hat{a}_z$$

$$= 2\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z$$

Example - 1.4 Three field quantities are given by $\vec{P} = 2\hat{a}_x - \hat{a}_z$ and $\vec{Q} = 2\hat{a}_x - \hat{a}_y + 2\hat{a}_z$, $\vec{R} = 2\hat{a}_x - 3\hat{a}_y + \hat{a}_z$. The value of $\vec{Q}\cdot(\vec{R}\times\vec{P})$ is

(a) 10 (b) 18

(c) 2 (d) 14

Solution: (d)

$$\vec{Q}.(\vec{R}\times\vec{P}) = (2, -1, 2)\,.\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= (2, -1, 2).\,(3, 4, 6)$$

$$= 6 - 4 + 12 = 14$$

Alternatively:

$$\vec{Q}.(\vec{R}\times\vec{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 14$$

Student's Assignment

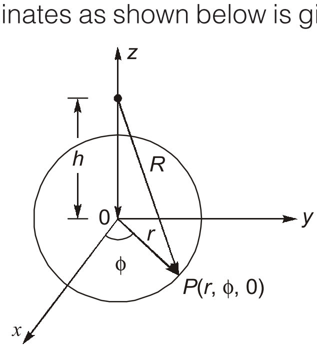
Q.1 A point is represented in Cartesian coordinates as $P(3, 4, 5)$, the radial component ρ in cylindrical coordinates will be _______ r in spherical coordinates.
(a) less than (b) greater than
(c) equal to (d) unrelated to

Q.2 Consider a vector $\vec{E} = z\hat{a}_x + (x+y)\hat{a}_y$, the z component of the vector in cylindrical coordinates will be
(a) z
(b) $z \cos\phi + (x + y) \sin\phi$
(c) $-z \sin\phi + (x + y) \cos\phi$
(d) zero

Q.3 Let a point in spherical and cylindrical coordinates are (r, θ, ϕ) and (ρ, ϕ, z). The radial component r in spherical coordinates is related to components in cylindrical coordinates as
(a) ρ (b) $\rho \cos\phi$
(c) $z \tan^{-1}\phi$ (d) $(\rho^2 + z^2)^{1/2}$

Q.4 Given the vector
$\vec{A} = (\cos x)(\sin y)\hat{a}_x + (\sin x)(\cos y)\hat{a}_y$,
where $\hat{a}_x, \hat{a}_y$ denote unit vectors along x, y directions, respectively. The magnitude of curl of $\vec{A}$ is
(a) 0 (b) 1
(c) –1 (d) 2

Q.5 The unit vector $\vec{a}_R$ which points from $z = h$ on the z-axis towards $(r, \phi, 0)$ in cylindrical co-ordinates as shown below is given by



(a) $\dfrac{h\vec{a}_r - r\vec{a}_z}{\sqrt{r^2+h^2}}$ (b) $\dfrac{r\vec{a}_r - h\vec{a}_z}{\sqrt{r^2+h^2}}$
(c) $\dfrac{h\vec{a}_\phi - r\vec{a}_z}{\sqrt{r^2+h^2}}$ (d) $\dfrac{r\vec{a}_z - h\vec{a}_\phi}{\sqrt{r^2+h^2}}$

Q.6 The vector field given by
$$\vec{A} = yz\vec{a}_x + xz\vec{a}_y + xy\vec{a}_z \text{ is}$$
(a) rotational and solenoidal
(b) rotational but not solenoidal
(c) irrotational and solenoidal
(d) irrotational but not solenoidal

Q.7 If $\vec{A} = \dfrac{\vec{a}_x}{\sqrt{x^2+y^2}}$, then the value of $\nabla \cdot \vec{A}$ at (2, 2, 0) will be
(a) –0.0884 (b) 0.0264
(c) –0.0356 (d) 0.0542

Q.8 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then the value of
$\vec{i} \times (\vec{r} \times \vec{i}) + \vec{j} \times (\vec{r} \times \vec{j}) + \vec{k} \times (\vec{r} \times \vec{k})$ is
(a) $\vec{r}$ (b) $2\vec{r}$
(c) $3\vec{r}$ (d) $6\vec{r}$

Q.9 What is the value of constant b so that the vector
$$\vec{V} = (x+3y)\vec{i} + (y-2x)\vec{j} + (x+bz)\vec{k}$$
is solenoidal?
(a) 2 (b) –1
(c) 3 (d) –2

Q.10 If $\vec{r} = x\vec{I}_x + y\vec{I}_y + z\vec{I}_z$, then which of the following relation will hold true?
(a) $\nabla\vec{r} = 3$
(b) $\nabla \times \vec{r} = 0$
(c) Both (a) and (b)
(d) Neither (a) nor (b)

Q.11 If $\vec{c} = \vec{a} \times \vec{b}$ and $\vec{b} = \vec{a} \times \vec{c}$, then

(a) $\vec{b} = 0$ and $\vec{c} = 0$ (b) Only $\vec{b} = 0$

(c) Only $\vec{c} = 0$ (d) $\vec{b} \neq 0$ and $\vec{c} \neq 0$

Q.12 If S is any closed surface enclosing a volume V and $\vec{A} = ax\vec{i} + by\vec{j} + cz\vec{k}$, then the value of $\iint_S \vec{A} \cdot \hat{n}\, d\vec{s}$ ($\hat{n}$ is a unit vector) will be equal to

(a) $\frac{1}{3}(a+b+c)V$ (b) $(a-b-c)V$

(c) $\frac{1}{2}(a+b+c)V$ (d) $(a+b+c)V$

ANSWER KEY — STUDENT'S ASSIGNMENT

1. (a)	**2.** (d)	**3.** (d)	**4.** (c)	**5.** (b)
6. (c)	**7.** (a)	**8.** (b)	**9.** (d)	**10.** (c)
11. (a)	**12.** (d)			

HINTS & SOLUTIONS — STUDENT'S ASSIGNMENT

5. (b)

Let the unit vector be given by $\vec{a}_R$.

Now, $\vec{R}$ = Difference of two vectors

$= r\vec{a}_r - h\vec{a}_z$

$\therefore$ Unit vector, $\vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{r\vec{a}_r - h\vec{a}_z}{\sqrt{r^2+h^2}}$

6. (c)

The vector field $\vec{A}$ will be irrotational, if $\nabla \times \vec{A} = 0$.

$$\text{Now, } \nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right]\vec{a}_x + \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right]\vec{a}_y + \left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right]\vec{a}_z$$

$$= [x-x]\vec{a}_x + [y-y]\vec{a}_y + [z-z]\vec{a}_z = 0$$

Hence, $\vec{A}$ is irrotational.

The vector field $\vec{A}$ will be solenoidal, if $\nabla \cdot \vec{A} = 0$

Here,

$$\nabla \cdot \vec{A} = \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}\right) \cdot (yz\vec{a}_x + xz\vec{a}_y + xy\vec{a}_z)$$

$$= \vec{a}_x \cdot \vec{a}_x \frac{\partial}{\partial x}(yz) + \vec{a}_y \cdot \vec{a}_y \frac{\partial}{\partial y}(xz) + \vec{a}_z \cdot \vec{a}_z \frac{\partial}{\partial z}(xy)$$

$$= 0 + 0 + 0 = 0$$

7. (a)

Given, $\vec{A} = \frac{1}{\sqrt{x^2+y^2}}\vec{a}_x$

$$\therefore \quad \nabla \cdot \vec{A} = \frac{\partial}{\partial x}(A_x) + \frac{\partial}{\partial y}(A_y) + \frac{\partial}{\partial z}(A_z)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2}}\right) + 0 + 0$$

$$= \frac{\partial}{\partial x}(x^2+y^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2+y^2)^{-3/2} \cdot 2x$$

$$= \nabla \cdot \vec{A} = -\frac{x}{\sqrt{(x^2+y^2)(x^2+y^2)}}$$

$$\text{Now, } (\nabla \cdot \vec{A})_{2,2,0} = -\frac{2}{\sqrt{(2^2+2^2)\cdot(2^2+2^2)}}$$

$$= -\frac{2}{\sqrt{8.8}} = -0.0884$$

8. (b)

Given, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore$ $\vec{r} \times \vec{i} = (x\vec{i} + y\vec{j} + z\vec{k}) \times \vec{i}$

$= -y\vec{k} + z\vec{j}$

Also, $\vec{i} \times (\vec{r} \times \vec{i}) = \vec{i} \times (-y\vec{k} + z\vec{j})$

$= \vec{j}y + z\vec{k}$

Similarly, $\vec{j} \times (\vec{r} \times \vec{j}) = \vec{i}x + \vec{k}z$

and $\vec{k} \times (\vec{r} \times \vec{k}) = \vec{i}x + \vec{j}y$

Thus, $\vec{i} \times (\vec{r} \times \vec{i}) + \vec{j} \times (\vec{r} \times \vec{j}) + \vec{k} \times (\vec{r} \times \vec{k})$

$= 2(x\vec{i} + y\vec{j} + z\vec{k}) = 2\vec{r}$