Thoroughly Revised and Updated

Engineering Mathematics

For

GATE 2020
and ESE 2020 Prelims

Comprehensive Theory with Solved Examples
Including **Previous Solved Questions of**
GATE (2003-2019) and **ESE-Prelims** (2017-2019)

Note: *Syllabus of ESE Mains Electrical Engineering also covered*

Engineering Mathematics for GATE 2020 and ESE 2020 Prelims

Preface

Over the period of time the GATE and ESE examination have become more challenging due to increasing number of candidates. Though every candidate has ability to succeed but competitive environment, in-depth knowledge, quality guidance and good source of study is required to achieve high level goals.

B. Singh (Ex. IES)

The new edition of **Engineering Mathematics for GATE 2020 and ESE 2020 Prelims** has been fully revised, updated and edited. The whole book has been divided into topicwise sections.

I have true desire to serve student community by way of providing good source of study and quality guidance. I hope this book will be proved an important tool to succeed in GATE and ESE examination. Any suggestions from the readers for the improvement of this book are most welcome.

B. Singh (Ex. IES)
Chairman and Managing Director
MADE EASY Group

SYLLABUS

GATE and ESE Prelims: Civil Engineering

Linear Algebra: Matrix algebra; Systems of linear equations; Eigen values and Eigen vectors.

Calculus: Functions of single variable; Limit, continuity and differentiability; Mean value theorems, local maxima and minima, Taylor and Maclaurin series; Evaluation of definite and indefinite integrals, application of definite integral to obtain area and volume; Partial derivatives; Total derivative; Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Ordinary Differential Equation (ODE): First order (linear and non-linear) equations; higher order linear equations with constant coefficients; Euler-Cauchy equations; Laplace transform and its application in solving linear ODEs; initial and boundary value problems.

Partial Differential Equation (PDE): Fourier series; separation of variables; solutions of one-dimensional diffusion equation; first and second order one-dimensional wave equation and two-dimensional Laplace equation.

Probability and Statistics: Definitions of probability and sampling theorems; Conditional probability; Discrete Random variables: Poisson and Binomial distributions; Continuous random variables: normal and exponential distributions; Descriptive statistics - Mean, median, mode and standard deviation; Hypothesis testing.

Numerical Methods: Accuracy and precision; error analysis. Numerical solutions of linear and non-linear algebraic equations; Least square approximation, Newton's and Lagrange polynomials, numerical differentiation, Integration by trapezoidal and Simpson's rule, single and multi-step methods for first order differential equations.

GATE and ESE Prelims: Mechanical Engineering

Linear Algebra: Matrix algebra, systems of linear equations, eigenvalues and eigenvectors.

Calculus: Functions of single variable, limit, continuity and differentiability, mean value theorems, indeterminate forms; evaluation of definite and improper integrals; double and triple integrals; partial derivatives, total derivative, Taylor series (in one and two variables), maxima and minima, Fourier series; gradient, divergence and curl, vector identities, directional derivatives, line, surface and volume integrals, applications of Gauss, Stokes and Green's theorems.

Differential equations: First order equations (linear and nonlinear); higher order linear differential equations with constant coefficients; Euler-Cauchy equation; initial and boundary value problems; Laplace transforms; solutions of heat, wave and Laplace's equations.

Complex Variables: Analytic functions; Cauchy-Riemann equations; Cauchy's integral theorem and integral formula; Taylor and Laurent series.

Probability and Statistics: Definitions of probability, sampling theorems, conditional probability; mean, median, mode and standard deviation; random variables, binomial, Poisson and normal distributions.

Numerical Methods: Numerical solutions of linear and non-linear algebraic equations; integration by trapezoidal and Simpson's rules; single and multi-step methods for differential equations..

GATE and ESE Prelims: Electrical Engineering

Linear Algebra: Matrix Algebra, Systems of linear equations, Eigenvalues, Eigenvectors.

Calculus: Mean value theorems, Theorems of integral calculus, Evaluation of definite and improper integrals, Partial Derivatives, Maxima and minima, Multiple integrals, Fourier series, Vector identities, Directional derivatives, Line integral, Surface integral, Volume integral, Stokes's theorem, Gauss's theorem, Green's theorem.

Differential equations: First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Method of variation of parameters, Cauchy's equation, Euler's equation, Initial and boundary value problems, Partial Differential Equations, Method of separation of variables.

Complex Variables: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula, Taylor series, Laurent series, Residue theorem, Solution integrals.

Probability and Statistics: Sampling theorems, Conditional probability, Mean, Median, Mode, Standard Deviation, Random variables, Discrete and Continuous distributions, Poisson distribution, Normal distribution, Binomial distribution, Correlation analysis, Regression analysis.

Numerical Methods: Solutions of nonlinear algebraic equations, Single and Multi-step methods for differential equations.

Transform Theory: Fourier Transform, Laplace Transform, z-Transform.

Electrical Engineering ESE Mains

Matrix theory, Eigen values & Eigen vectors, system of linear equations, Numerical methods for solution of non-linear algebraic equations and differential equations, integral calculus, partial derivatives, maxima and minima, Line, Surface and Volume Integrals. Fourier series, linear, nonlinear and partial differential equations, initial and boundary value problems, complex variables, Taylor's and Laurent's series, residue theorem, probability and statistics fundamentals, Sampling theorem, random variables, Normal and Poisson distributions, correlation and regression analysis.

GATE and ESE Prelims: Electronics Engineering

Linear Algebra: Vector space, basis, linear dependence and independence, matrix algebra, eigen values and eigen vectors, rank, solution of linear equations – existence and uniqueness.

Calculus: Mean value theorems, theorems of integral calculus, evaluation of definite and improper integrals, partial derivatives, maxima and minima, multiple integrals, line, surface and volume integrals, Taylor series.

Differential equations: First order equations (linear and nonlinear), higher order linear differential equations, Cauchy's and Euler's equations, methods of solution using variation of parameters, complementary function and particular integral, partial differential equations, variable separable method, initial and boundary value problems.

Vector Analysis: Vectors in plane and space, vector operations, gradient, divergence and curl, Gauss's, Green's and Stoke's theorems.

Complex Analysis: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula; Taylor's and Laurent's series, residue theorem.

Numerical Methods: Solution of nonlinear equations, single and multi-step methods for differential equations, convergence criteria.

Probability and Statistics: Mean, median, mode and standard deviation; combinatorial probability, probability distribution functions - binomial, Poisson, exponential and normal; Joint and conditional probability; Correlation and regression analysis.

GATE: Instrumentation Engineering

Linear Algebra : Matrix algebra, systems of linear equations, Eigen values and Eigen vectors.

Calculus : Mean value theorems, theorems of integral calculus, partial derivatives, maxima and minima, multiple integrals, Fourier series, vector identities, line, surface and volume integrals, Stokes, Gauss and Green's theorems.

Differential Equations : First order equation (linear and nonlinear), higher order linear differential equations with constant coefficients, method of variation of parameters, Cauchy's and Euler's equations, initial and boundary value problems, solution of partial differential equations: variable separable method.

Analysis of complex variables: : Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent's series, residue theorem, solution of integrals.

Complex Variables : Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent' series, Residue theorem, solution integrals.

Probability and Statistics : Sampling theorems, conditional probability, mean, median, mode and standard deviation, random variables, discrete and continuous distributions: normal, Poisson and binomial distributions.

Numerical Methods : Matrix inversion, solutions of non-linear algebraic equations, iterative methods for solving differential equations, numerical integration, regression and correlation analysis.

GATE: Computer Science & IT Engineering

Linear Algebra: Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LU decomposition.

Calculus: Limits, continuity and differentiability. Maxima and minima. Mean value theorem. Integration.

Probability: Random variables. Uniform, normal, exponential, poisson and binomial distributions. Mean, median, mode and standard deviation. Conditional probability and Bayes theorem.

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1 Linear Algebra

1.1 Introduction

In this chapter, we shall discuss matrix algebra and its use in solving linear system of algebraic equations $AX = B$ and in solving the Eigen value problem $AX = \lambda X$.

1.2 Algebra of Matrices

1.2.1 Definition of Matrix

A system of $m \times n$ numbers arranged in the form of a rectangular array having m rows and n columns is called an matrix of order $m \times n$.

If $A = [a_{ij}]_{m \times n}$ be any matrix of order $m \times n$ then it is written in the form:

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12}...............a_{1n} \\ a_{21} & a_{22}...............a_{2n} \\ .... & .... \\ .... & .... \\ a_{m1} & a_{m2}...............a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

1.2.2 Special Types of Matrices

1. **Square Matrix:** An $m \times n$ matrix for which $m = n$ (The number of rows is equal to number of columns) is called square matrix. It is also called an n-rowed square matrix. i.e. $A = [a_{ij}]_{n \times n}$. The elements $a_{ij} \mid i = j$, i.e. a_{11}, a_{22}.... are called **DIAGONAL ELEMENTS** and the line along which they lie is called **PRINCIPLE DIAGONAL** of matrix. Elements other than a_{11}, a_{22}, etc are called off-diagonal elements i.e. $a_{ij} \mid i \neq j$.

    Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 3 \end{bmatrix}_{3 \times 3}$ is a square Matrix

> **NOTE**
>
> A square sub-matrix of a square matrix A is called a **"principle sub-matrix"** if its diagonal elements are also the diagonal elements of the matrix A. So $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ is a principle sub matrix of the matrix A given above, but $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ is not.

2. **Diagonal Matrix:** A square matrix in which all off-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} & \text{if } i = j \end{cases}$

Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is a diagonal matrix

The above matrix can also be written as $A = $ diag [3, 5, 9]

Properties of Diagonal Matrix:

diag $[x, y, z] + $ diag $[p, q, r] = $ diag $[x + p, y + q, z + r]$

diag $[x, y, z] \times $ diag $[p, q, r] = $ diag $[xp, yq, zr]$

(diag $[x, y, z])^{-1} = $ diag $[1/x, 1/y, 1/z]$

(diag $[x, y, z])^T = $ diag $[x, y, z]$

(diag $[x, y, z])^n = $ diag $[x^n, y^n, z^n]$

Eigen values of diag $[x, y, z] = x, y$ and z.

Determinant of diag $[x, y, z] = |$ diag $[x, y, z]| = xyz$

3. **Scalar Matrix:** A scalar matrix is a diagonal matrix with all diagonal elements being equal.

$\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} = k & \text{if } i = j \end{cases}$

Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix.

4. **Unit Matrix or Identity Matrix:** A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I. Identity matrix is always square.

Thus a square matrix $A = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$. $\begin{cases} a_{ij} = 0 & \text{if } i \neq j \\ a_{ij} = 1 & \text{if } i = j \end{cases}$

Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Properties of Identity Matrix:

(a) I is Identity element for multiplication, so it is called multiplicative identity.

(b) $AI = IA = A$

(c) $I^n = I$

(d) $I^{-1} = I$

(e) $|I| = 1$

5. **Null Matrix:** The $m \times n$ matrix whose elements are all zero is called null matrix.

Null matrix is denoted by O. Null matrix need not be square. $a_{ij} = 0 \ \forall \ i, j$

Example: $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{2 \times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Properties of Null Matrix:

(a) $A + O = O + A = A$

  So, O is additive identity.

(b) $A + (-A) = O$

6. **Upper Triangular Matrix:** An upper triangular matrix is a square matrix whose lower off-diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i > j$

   It is denoted by U.

   The diagonal and upper off diagonal elements may or may not be zero. $\begin{cases} a_{ij} = 0 & \text{if } i > j \\ a_{ij} & \text{if } i < j \end{cases}$

   Example: $U = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

7. **Lower Triangular Matrix:** A lower triangular matrix is a square matrix whose upper off-diagonal triangular elements are zero, i.e. $a_{ij} = 0$ whenever $i < j$. The diagonal and lower off-diagonal elements may or may not be zero. $\{a_{ij} = 0 \quad \text{if } i < j$

   It is denoted by L,

   Example: $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & 3 & 6 \end{bmatrix}$

8. **Idempotent Matrix:** A matrix A is called Idempotent if $A^2 = A$.

   Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are examples of Idempotent matrices.

9. **Involuntary Matrix:** A matrix A is called Involutory if $A^2 = I$.

   Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is involuntary. Also $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is involuntary since $A^2 = I$.

10. **Nilpotent Matrix:** A matrix A is said to be nilpotent of class x or index x if $A^x = O$ and $A^{x-1} \neq O$ i.e. x is the smallest index which makes $A^x = O$.

    Example: The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since $A \neq 0$ and $A^2 \neq 0$, but $A^3 = 0$.

11. **Singular matrix:** If the determinant of a matrix is zero, then matrix is called as singular matrix.

    $|A| = 0$ e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

    If determinant is not zero, then matrix is known as non-singular matrix.

    If matrix is singular then its inverse doesn't exist.

12. **Row Matrix:** A matrix with only one row is called as Row Matrix. A row matrix can have any number of columns i.e., it has an order $1 \times n$ where $n \in$ natural number.

    Example: (i)   [1, 2, -7, 0, 4] is a row matrix of order 1×5.

                  (ii)   [3, 0, -5] is a row matrix of order 1×3

13. **Column Matrix:** A matrix with only one column is called as column matrix. A column matrix can have any number of rows i.e., it has an order $m \times 1$ where $m \in$ natural number.

Example: (i) $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ is a column matrix of order 3 × 1. (ii) $\begin{bmatrix} 1 \\ 7 \\ -\dfrac{7}{3} \end{bmatrix}$ is a column matrix of order 2 × 1.

1.2.3 Equality of Two Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if,

1.   They are of same size.
2.   The elements in the corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Example: Let $\begin{bmatrix} x-y & p+q \\ p-q & x+y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$

Then $x - y = 2$, $p + q = 5$, $p - q = 1$ and $x + y = 10$

$\Rightarrow$ $x = 6$, $y = 4$, $p = 3$ and $q = 2$.

1.2.4   Addition of Matrices

Two matrices A and B are compatible for addition only if they both have exactly the same size say $m \times n$. Then their sum is defined to be the matrix of the type $m \times n$ obtained by adding corresponding elements of A and B. Thus if, $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$;

$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$

Properties of Matrix Addition:

1.   Matrix addition is commutative $A + B = B + A$.
2.   Matrix addition is associative $(A + B) + C = A + (B + C)$
3.   Existence of additive identity: If O be $m \times n$ matrix each of whose elements are zero. Then, $A + O = A = O + A$ for every $m \times n$ matrix A.
4.   Existence of additive inverse: Let $A = [a_{ij}]_{m \times n}$.

     Then the negative of matrix A is defined as matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$.

     $\Rightarrow$ Matrix $-A$ is additive inverse of A. Because $(-A) + A = O = A + (-A)$. Here O is null matrix of order $m \times n$.

5.   Cancellation laws holds good in case of addition of matrices of same order.

     $A + X = B + X \Rightarrow A = B$

     $X + A = X + B \Rightarrow A = B$

     Example: Let A, B, C are matrices of same order i.e., $m \times n$ then, $A + B = A + C$ holds only if $B = C$.

6.   The equation $A + X = 0$ has a unique solution in the set of all $m \times n$ matrices.

1.2.5 Substraction of Two Matrices

If A and B are two $m \times n$ matrices, then we define, $A - B = A + (-B)$.

Thus the difference $A - B$ is obtained by subtracting from each element of A corresponding elements of B.

NOTE: Subtraction of matrices is neither commutative nor associative.

1.2.6 Multiplication of a Matrix by a Scalar

Let A be any $m \times n$ matrix and k be any real number called scalar. The $m \times n$ matrix obtained by multiplying every element of the matrix A by k is called scalar multiple of A by k and is denoted by kA.

$\Rightarrow$ If $A = [a_{ij}]_{m \times n}$ then $Ak = kA = [kA]_{m \times n}$.

If $A = \begin{bmatrix} 5 & 2 & 1 \\ 6 & -5 & 2 \\ 1 & 3 & 6 \end{bmatrix}$ then, $3A = \begin{bmatrix} 15 & 6 & 3 \\ 18 & -15 & 6 \\ 3 & 9 & 18 \end{bmatrix}$

Properties of Multiplication of a Matrix by a Scalar:

1. Scalar multiplication of matrices distributes over the addition of matrices i.e., $k(A + B) = kA + kB$.
2. If p and q are two scalars and A is any $m \times n$ matrix then, $(p + q)A = pA + qA$.
3. If p and q are two scalers and $A = [a_{ij}]_{m \times n}$ then, $p(qA) = (pq)A$.
4. If $A = [a_{ij}]_{m \times n}$ be a matrix and k be any scalar then, $(-k)A = -(kA) = k(-A)$.

1.2.7 Multiplication of Two Matrices

Let $A = [a_{ij}]_{m \times n}$; $B = [b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [c_{ik}]_{m \times p}$ such that $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$ is called the product of matrices A and B in that order and we write $C = AB$.

Properties of Matrix Multiplication:

1. Multiplication of matrices is not commutative. In fact, if the product of AB exists, then it is not necessary that the product of BA will also exist. For example, $A_{3 \times 2} \times B_{2 \times 4} = C_{3 \times 4}$ but $B_{2 \times 4} \times A_{3 \times 2}$ does not exist since these are not compatible for multiplication.

Example:

Show that $AB \neq BA$ if,

$$A = \begin{bmatrix} 1 & 7 & -9 \\ -8 & 4 & 3 \\ 0 & 1 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 3 & -5 \\ 4 & 0 & 5 \end{bmatrix}$$

Solution:

$$AB = \begin{bmatrix} 1 \times 4 + 7 \times 1 - 9 \times 4 & 1 \times 0 + 7 \times 3 - 9 \times 0 & 1 \times -2 + 7 \times (-5) - 9 \times 5 \\ -8 \times 4 + 4 \times 1 + 3 \times 4 & -8 \times 0 + 4 \times 3 + 3 \times 0 & 0 \times -2 + 1 \times -5 + -2 \times 5 \\ 0 \times 4 + 1 \times 1 - 2 \times 4 & 0 \times 0 + 1 \times 3 - 2 \times 0 & 0 \times -2 + 1 \times -5 - 2 \times 5 \end{bmatrix}$$

$\Rightarrow \qquad AB = \begin{bmatrix} -25 & 21 & -82 \\ -16 & 12 & 11 \\ -7 & 3 & -15 \end{bmatrix} \qquad ...(1)$

Similarly, $\qquad BA = \begin{bmatrix} 4 & 26 & -32 \\ -23 & 14 & 10 \\ 4 & 33 & -46 \end{bmatrix} \qquad ...(2)$

from (1) and (2) $\quad AB \neq BA$

2. Matrix multiplication is associative, if conformability is assured. i.e., $A(BC) = (AB)C$ where A, B, C are $m \times n$, $n \times p$, $p \times q$ matrices respectively.

3. Multiplication of matrices is distributive with respect to addition of matrices. i.e., $A(B + C) = AB + AC$.

4. The equation $AB = O$ does not necessarily imply that at least one of matrices A and B must be a zero matrix. For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

5. In the case of matrix multiplication if $AB = O$ then it is not necessarily imply that $BA = O$. In fact, BA may not even exist.

6. Both left and right cancellation laws hold for matrix multiplication as shown below:

    $AB = AC \Rightarrow B = C$ (if A is non-singular matrix) and

    $BA = CA \Rightarrow B = C$ (if A is non-singular matrix).

7. Product AA exists only when A is a square matrix.

    If m and n are two numbers and A is a square matrix then,

$$A^m A^n = A^{m+n}$$

8. If A is a square matrix of order 'n' and I_n is an identity matrix of order 'n' then,

$$AI_n = I_n A = A$$

1.2.8 Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by $Tr(A)$.

Thus if $A = [a_{ij}]_{n \times n}$ then, $Tr(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \dots a_{nn}$.

Let
$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{bmatrix}$$

Then, Trace $(A) = Tr(A) = 1 + (-3) + 5 = 3$

Properties of Trace of a Matrix:

Let A and B be two square matrices of order n and λ be a scalar. Then,

1. $Tr(\lambda A) = \lambda\ Tr\ A$

2. $Tr(A + B) = Tr\ A + Tr\ B$

3. $Tr(AB) = Tr(BA)$ [If both AB and BA are defined]

1.2.9 Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A' or A^T.

Let
$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix} \text{ then, } A^T = A' = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$$

If
$$B = [1\ 2\ 3]$$

Then
$$B' = [1\ 2\ 3]' = [1\ 2\ 3]^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of Transpose of a Matrix:

If A^T and B^T be transposes of A and B respectively then,

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$, k being any complex number
4. $(AB)^T = B^T A^T$
5. $(ABC)^T = C^T B^T A^T$

1.2.10 Conjugate of a Matrix

The matrix obtained from given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$.

Example: If $A = \begin{bmatrix} 2 + 3i & 4 - 7i & 8 \\ -i & 6 & 9 + i \end{bmatrix}$

$$\overline{A} = \begin{bmatrix} 2 - 3i & 4 + 7i & 8 \\ +i & 6 & 9 - i \end{bmatrix}$$

Properties of Conjugate of a Matrix:

If $\overline{A}$ and $\overline{B}$ be the conjugates of A and B respectively. Then,

1. $\overline{(\overline{A})} = A$
2. $\overline{(A + B)} = \overline{A} + \overline{B}$
3. $\overline{(kA)} = \overline{k}\,\overline{A}$, k being any complex number
4. $\overline{(AB)} = \overline{A}\,\overline{B}$, A and B being conformable to multiplication
5. $\overline{A} = A$ if A is real matrix
    $\overline{A} = -A$ if A is purely imaginary matrix

1.2.11 Transposed Conjugate of Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ or $A^\star$ or $(\overline{A})^T$. It is also called conjugate transpose of A. This is also known as tranjugate of A.

Example: If $A = \begin{bmatrix} 2 + i & 3 - i \\ 4 & 1 - i \end{bmatrix}$

To find A^θ, we first find $\overline{A} = \begin{bmatrix} 2 - i & 3 + i \\ 4 & 1 + i \end{bmatrix}$

Then $A^\theta = (\overline{A})^T = \begin{bmatrix} 2 - i & 4 \\ 3 + i & 1 + i \end{bmatrix}$

Some properties: If A^θ & B^θ be the transposed conjugates of A and B respectively then,

1. $(A^\theta)^\theta = A$
2. $(A + B)^\theta = A^\theta + B^\theta$
3. $(kA)^\theta = \overline{k}A^\theta$, $k \rightarrow$ complex number
4. $(AB)^\theta = B^\theta A^\theta$

1.2.12 Classification of Real Matrices

Real matrices can be classified into the following three types based on the relationship between A^T and A.

1. Symmetric Matrices ($A^T = A$)
2. Skew Symmetric Matrices ($A^T = -A$)
3. Orthogonal Matrices ($A^T = A^{-1}$ or $AA^T = I$)

1. **Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ elements is same as its $(j, i)^{th}$ element i.e., $a_{ij} = a_{ji}$ for all i & j.

In a symmetric matrix, $A^T = A$

Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix, since $A^T = A$.

Note: For any matrix A,

(a) AA^T is always a symmetric matrix.

(b) $\dfrac{A + A^T}{2}$ is always symmetric matrix.

   Note: If A and B an symmetric, then

(a) $A + B$ and $A - B$ are also symmetric.

(b) AB, BA may or may not be symmetric.

2. **Skew Symmetric Matrix:** A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $(i, j)^{th}$ elements of A is the negative of the $(j, i)^{th}$ elements of A if $a_{ij} = -a_{ji}$ $\forall$ i, j.

In a skew symmetric matrix $A^T = -A$.

A skew symmetric matrix must have all 0's in the diagonal.

Example: $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

   Note: For any matrix A, the matrix $\dfrac{A - A^T}{2}$ is always skew symmetric.

Every square matrix can be uniquely expressed as sum of a symmetrical and skew symmetric matrix.

Example: Let any matrix A of order $n \times n$ then,

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2} = P + Q$$

where, $P \to$ Symmetric matrix and $Q \to$ Skew symmetric matrix

$$P = \frac{A + A^T}{2}, \quad Q = \frac{A - A^T}{2}$$

3. **Orthogonal Matrix:** A square matrix A is said be orthogonal if:

$A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. Thus A will be an orthogonal matrix if, $AA^T = I = A^TA$.

Example: The identity matrix is orthogonal since $I^T = I^{-1} = I$.

Note: Since for an orthogonal matrix A,

$$AA^T = I$$
$$\Rightarrow \quad |AA^T| = |I| = 1$$
$$\Rightarrow \quad |A||A^T| = 1$$
$$\Rightarrow \quad (|A|)^2 = 1$$
$$\Rightarrow \quad |A| = \pm 1$$

So the determinant of an orthogonal matrix always has a modulus of 1.

Note:

(i)   If A and B are orthogonal matrices, then AB and BA are also orthogonal.

(ii) If matrices A is orthogonal then Det $(A) = \pm 1$ but if a matrices A has Det $(A) = \pm 1$ then it may or may not be orthogonal.

1.2.13 Classification of Complex Matrices

Complex matrices can be classified into the following three types based on relationship between A^θ and A.

1. Hermitian Matrix $(A^\theta = A)$ 2. Skew-Hermitian Matrix $(A^\theta = -A)$ 3. Unitary Matrix $(A^\theta = A^{-1}$ or $AA^\theta = I)$

1. **Hermitian Matrix:** A necessary and sufficient condition for a matrix A to be Hermitian is that $A^\theta = A$.

    Example: $A = \begin{bmatrix} a & b + ic \\ b - ic & d \end{bmatrix}$ is a Hermitian matrix.

2. **Skew-Hermitian Matrix:** A necessary and sufficient condition for a matrix to be skew-Hermitian if $A^\theta = -A$.

    Example: $A = \begin{bmatrix} 0 & -2 - i \\ 2 - i & 0 \end{bmatrix}$ is skew-Hermitian.

3. **Unitary Matrix:** A square matrix A is said to be unitary if:
    $$A^\theta = A^{-1}$$
    Multiplying both sides by A, we get an alternate definition of unitary matrix as given below:

    A square matrix A is said to be unitary if:
    $$AA^\theta = I = A^\theta A$$

    Example: $A = \begin{bmatrix} \dfrac{1+i}{2} & \dfrac{-1+i}{2} \\ \dfrac{1+i}{2} & \dfrac{1-i}{2} \end{bmatrix}$ is an example of a unitary matrix.

1.2.14 Properties of Complex Matrices

(i) $(A + B)^\theta = A^\theta + B^\theta$

(ii) $(AB)^\theta = B^\theta A^\theta$

(iii) The diagonal elements of a Hermitian matrices are necessarily real.

(iv)   The diagonal elements of a skew Hermitian matrices are necessarily either purely imaginary or zero.

(v)   Every square matrix A can be written as sum of Hermitian matrices P and skew Hermitian matrices Q.

$$A = \frac{A + A^\theta}{2} + \frac{A - A^\theta}{2} = P + Q \quad \text{Here,} \quad P \rightarrow \text{Herimitian matrix}$$
$$Q \rightarrow \text{Skew Hermitian matrix}$$

(vi)   Every square matrix A can be written in form of $P + iQ$ where P and Q are Hermitian matrix.

$$A = \left(\frac{A + A^\theta}{2}\right) + i\left(\frac{A - A^\theta}{2}\right) = P + iQ \quad \text{Here, } P \text{ and } Q \text{ are Hemitian matrices.}$$

(vii) Modulus of determinant of a unitary matrix is unity.

   Example: Let A is a unitary matrix then,
$$A^\theta A = I \quad \Rightarrow \quad \left| A^\theta A \right| = 1$$
$$\Rightarrow \quad \left| A^\theta \right| \left| A \right| = 1 \quad \Rightarrow \quad \left| (\overline{A})^T \right| \left| A \right| = 1 \quad \Rightarrow \quad \left| A^T \right| \left| A \right| = 1 \quad \Rightarrow \quad \left| A \right|^2 = 1$$
$$\Rightarrow \quad \left| A \right| = \pm 1 \quad \Rightarrow \quad \| A \| = 1$$

Previous GATE and ESE Questions

Q.1 Given Matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is
(a) 4   (b) 3
(c) 2   (d) 1
[CE, GATE-2003, 1 mark]

Q.2 Consider the system of simultaneous equations
$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$
This system has
(a) unique solution
(b) infinite number of solutions
(c) no solution
(d) exactly two solutions
[ME, GATE-2003, 2 marks]

Q.3 Consider the following system of linear equations
$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$
Notice that the second and the third columns of the coefficient matrix are linearly dependent. For how many values of α, does this system of equations have infinitely many solutions?
(a) 0   (b) 1
(c) 2   (d) infinitely many
[CS, GATE-2003, 2 marks]

Q.4 For the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ the eigen values are
(a) 3 and –3   (b) –3 and –5
(c) 3 and 5   (d) 5 and 0
[ME, GATE-2003, 1 mark]

Q.5 For which value of x will the matrix given below become singular?
$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$
(a) 4   (b) 6
(c) 8   (d) 12
[ME, GATE-2004, 2 marks]

Q.6 Let A, B, C, D be $n \times n$ matrices, each with non-zero determinant, If $ABCD = I$, then B^{-1} is
(a) $D^{-1} C^{-1} A^{-1}$
(b) CDA
(c) ADC
(d) does not necessarily exist
[CS, GATE-2004, 1 mark]

Q.7 How many solutions does the following system of linear equations have?
$$-x + 5y = -1 \quad ; \quad x - y = 2 \quad ; \quad x + 3y = 3$$
(a) infinitely many   (b) two distinct solutions
(c) unique   (d) none
[CS, GATE-2004, 2 marks]

Q.8 The eigen values of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$
(a) are 1 and 4   (b) are –1 and 2
(c) are 0 and 5   (d) cannot be determined
[CE, GATE-2004, 2 marks]

Q.9 The sum of the eigen values of the matrix given below is $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.
(a) 5   (b) 7
(c) 9   (d) 18
[ME, GATE-2004, 1 mark]

Q.10 Consider the matrices $X_{(4 \times 3)}$, $Y_{(4 \times 3)}$ and $P_{(2 \times 3)}$. The order of $[P(X^TY)^{-1} P^T]^T$ will be
(a) (2×2) (b) (3×3)
(c) (4×3) (d) (3×4)
[CE, GATE-2005, 1 mark]

Q.11 Given an orthogonal matrix
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \left[AA^T \right]^{-1} \text{ is}$$

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Q.12 If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, then top row of R^{-1} is

(a) [5   6   4]   (b) [5   –3   1]
(c) [2   0   –1]   (d) [2   –1   1/2]

[EE, GATE-2005, 2 marks]

Q.13 Let, $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$.

Then $(a + b) =$

(a) $\dfrac{7}{20}$ (b) $\dfrac{3}{20}$

(c) $\dfrac{19}{60}$ (d) $\dfrac{11}{20}$

[EC, GATE-2005, 2 marks]

Q.14 Consider a non-homogeneous system of linear equations representing mathematically an over-determined system. Such a system will be
(a) consistent having a unique solution
(b) consistent having many solutions
(c) inconsistent having a unique solution
(d) inconsistent having no solution

[CE, GATE-2005, 1 mark]

Q.15 In the matrix equation $Px = q$, which of the following is a necessary condition for the existence of at least one solution for the unknown vector x
(a) Augmented matrix $[Pq]$ must have the same rank as matrix P

(b) Vector q must have only non-zero elements
(c) Matrix P must be singular
(d) Matrix P must be square

[EE, GATE-2005, 1 mark]

Q.16 Consider the following system of equations in three real variables x_1, x_2 and x_3
$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 - 2x_2 + 5x_3 = 2$$
$$-x_1 - 4x_2 + x_3 = 3$$
This system of equations has
(a) no solution
(b) a unique solution
(c) more than one but a finite number of solutions
(d) an infinite number of solutions

[CS, GATE-2005, 2 marks]

Q.17 Which one of the following is an eigen vector of

the matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

[ME, GATE-2005, 2 marks]

Q.18 For the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen

values is equal to –2. Which of the following is an eigen vector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

[EE, GATE-2005, 2 marks]

Q.19 Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, the eigen vector is

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Q.20 What are the eigen values of the following 2×2 matrix?

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

(a) -1 and 1 (b) 1 and 6
(c) 2 and 5 (d) 4 and -1

[CS, GATE-2005, 2 marks]

Q.21 Consider the system of equations $A_{(n \times n)} X_{(n \times 1)}$ $= \lambda_{(n \times 1)}$ where, λ is a scalar. Let (λ_i, x_i) be an eigen-pair of an eigen value and its corresponding eigen vector for real matrix A. Let I be a $(n \times n)$ unit matrix. Which one of the following statement is NOT correct?

(a) For a homogeneous $n \times n$ system of linear equations, $(A - \lambda I)x = 0$ having a nontrivial solution, the rank of $(A - \lambda I)$ is less than n

(b) For matrix A^m, m being a positive integer, (λ_i^m, x_i^m) will be the eigen-pair for all i

(c) If $A^T = A^{-1}$, then $|\lambda_i| = 1$ for all i

(d) If $A^T = A$, then λ_i is real for all i

[CE, GATE-2005, 2 marks]

Q.22 Multiplication of matrices E and F is G. Matrices E and G are

$$E \equiv \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What is the matrix F?

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[ME, GATE-2006, 2 marks]

Q.23 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I
A. Singular matrix
B. Non-square matrix
C. Real symmetric
D. Orthogonal matrix

List-II
1. Determinant is not defined
2. Determinant is always one
3. Determinant is zero
4. Eigen values are always real
5. Eigen values are not defined

Codes:

	A	B	C	D
(a)	3	1	4	2
(b)	2	3	4	1
(c)	3	2	5	4
(d)	3	4	2	1

[ME, GATE-2006, 2 marks]

Q.24 The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

[EC, GATE-2006, 1 mark]

Q.25 $P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T$, $Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T$ and $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$ are three vectors. An orthogonal set of vectors having a span that contains P, Q, R is

(a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

[EE, GATE-2006, 2 marks]

Q.26 The following vector is linearly dependent upon the solution to the previous problem

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

[EE, GATE-2006, 2 marks]

Q.27 Solution for the system defined by the set of equations $4y + 3z = 8$; $2x - z = 2$ and $3x + 2y = 5$ is
(a) $x = 0$; $y = 1$; $z = 4/3$
(b) $x = 0$; $y = 1/2$; $z = 2$
(c) $x = 1$; $y = 1/2$; $z = 2$
(d) non-existent

[CE, GATE-2006, 1 mark]

Q.28 For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

(a) 2   (b) 4
(c) 6   (d) 8

[EC, GATE-2006, 2 marks]

Q.29 For a given matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$, one of the eigen values is 3. The other two eigen values are
(a) 2, –5   (b) 3, –5
(c) 2, 5   (d) 3, 5

[CE, GATE-2006, 2 marks]

Q.30 Eigen values of a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1.

What are the eigen values of the matrix $S^2 = SS$?
(a) 1 and 25   (b) 6 and 4
(c) 5 and 1   (d) 2 and 10

[ME, GATE-2006, 2 marks]

Q.31 The eigen values and the corresponding eigen vectors of a 2×2 matrix are given by

     Eigen value   Eigen vector

     $\lambda_1 = 8$ $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

     $\lambda_2 = 4$ $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

[EC, GATE-2006, 2 marks]

Q.32 $[A]$ is square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and difference of these matrices are defined as $[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$, respectively. Which of the following statements is **TRUE**?
(a) Both $[S]$ and $[D]$ are symmetric
(b) Both $[S]$ and $[D]$ are skew-symmetric
(c) $[S]$ is skew-symmetric and $[D]$ is symmetric
(d) $[S]$ is symmetric and $[D]$ is skew-symmetric

[CE, GATE-2007, 1 mark]

Q.33 The inverse of the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is

(a) $\dfrac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$ (b) $\dfrac{1}{3}\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$

(c) $\dfrac{1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$ (d) $\dfrac{1}{3}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix}$

[CE, GATE-2007, 2 marks]

Q.34 $X = [x_1, x_2, \dots x_n]^T$ is an n-tuple nonzero vector. The $n \times n$ matrix $V = XX^T$
(a) has rank zero   (b) has rank 1
(c) is orthogonal   (d) has rank n

[EE, GATE-2007, 1 mark]

Q.35 It is given that $X_1, X_2, \dots X_M$ are M non-zero, orthogonal vectors. The dimension of the vector space spanned by the 2M vectors $X_1, X_2, \dots X_M$, $-X_1, -X_2, \dots -X_M$ is
(a) $2M$
(b) $M + 1$
(c) M
(d) dependent on the choice of $X_1, X_2, \dots X_M$

[EC, GATE-2007, 2 marks]

Q.36 Consider the set of (column) vectors defined by $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0$, where $x^T =[x_1, x_2, x_3]^T\}$. Which of the following is TRUE?

(a) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a basis for the subspace X.

(b) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set, but it does not span X and therefore is not a basis of X

(c) X is not a subspace for R^3

(d) None of the above

[CS, GATE-2007, 2 marks]

Q.37 For what values of α and β, the following simultaneous equations have an infinite number of solutions?

$$x + y + z = 5$$
$$x + 3y + 3z = 9$$
$$x + 2y + \alpha z = \beta$$

(a) 2, 7   (b) 3, 8
(c) 8, 3   (d) 7, 2

[CE, GATE-2007, 2 marks]

Q.38 The number of linearly independent eigen vectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

(a) 0   (b) 1
(c) 2   (d) infinite

[ME, GATE-2007, 2 marks]

Q.39 The linear operation $L(x)$ is defined by the cross product $L(x) = b \times X$, where b=$[0\ 1\ 0]^T$ and $X = [x_1\ x_2\ x_3]^T$ are three dimensional vectors. The 3×3 matrix M of this operation satisfies

$$L(x) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then the eigen values of M are
(a) 0, +1, −1   (b) 1, −1, 1
(c) $i, -i, 1$ (d) $i, -i, 0$

[EE, GATE-2007, 2 marks]

Q.40 The minimum and the maximum eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are −2 and 6, respectively.

What is the other eigen value?
(a) 5   (b) 3
(c) 1   (d) −1

[CE, GATE-2007, 1 mark]

Q.41 If a square matrix A is real and symmetric, then the eigen values

(a) are always real

(b) are always real and positive

(c) are always real and non-negative

(d) occur in complex conjugate pairs

[ME, GATE-2007, 1 mark]

Statement for Linked Answer Question 42 and 43.
Cayley-Hamilton Theorem states that a square matrix satisfies its own characteristic equation. Consider a matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

Q.42 A satisfies the relation
(a) $A + 3I + 2A^{-1} = 0$ (b) $A^2 + 2A + 2I =0$
(c) $(A + I) (A + 2I) = I$ (d) $\exp (A) = 0$

[EE, GATE-2007, 2 marks]

Q.43 A^9 equals
(a) $511\,A + 510I$ (b) $309\,A + 104I$
(c) $154\,A + 155I$ (d) $\exp (9A)$

[EE, GATE-2007, 2 marks]

Q.44 Let A be an $n \times n$ real matrix such that $A^2 = I$ and y be an n-dimensional vector.

Then the linear system of equations $Ax = y$ has
(a) no solution

(b) a unique solution

(c) more than one but finitely many independent solutions

(d) infinitely many independent solutions

[IN, GATE-2007 : 1 Mark]

Q.45 Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, $1 \leq i, j \leq n$, with $n \geq 3$ and $a_{ij} = i.j$.

Then the rank of A is
(a) 0   (b) 1
(c) $n - 1$ (d) n

[IN, GATE-2007 : 2 Marks]

Q.46 The determinant $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$ equals to

(a) 0   (b) $2b(b-1)$
(c) $2(1 - b)(1 + 2b)$ (d) $3b(1 + b)$

[PI, GATE-2007, 1 mark]

Q.47 If A is square symmetrical real valued matrix of dimensions $2n$, then eigen values of A are
(a) $2n$ distinct real values
(b) $2n$ real values not necessarily distinct
(c) n distinct pair of complex conjugate numbers
(d) n pairs of complex conjugate numbers not necessarily distinct.

[PI, GATE-2007, 2 marks]

Q.48 The inverse of matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

[PI, GATE-2008, 2 marks]

Q.49 The eigen vector pair of the matrix $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

[PI, GATE-2008, 2 marks]

Q.50 The product of matrices $(PQ)^{-1}P$ is
(a) P^{-1} (b) Q^{-1}
(c) $P^{-1} Q^{-1} P$ (d) $PQ\,P^{-1}$

[CE, GATE-2008, 1 mark]

Q.51 A is $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let matrix $A' = (A^TA)^{-1} A^T$, Then, which one of the following statement is TRUE?
(a) $AA'A = A$ (b) $(AA')^2 = A$
(c) $AA'A = I$ (d) $AA'A = A'$

[EE, GATE-2008, 2 marks]

Q.52 If the rank of a (5×6) matrix Q is 4, then which one of the following statements is correct?
(a) Q will have four linearly independent rows and four linearly independent columns
(b) Q will have four linearly independent rows and five linearly independent columns

(c) QQ^T will be invertible
(d) Q^TQ will be invertible

[EE, GATE-2008, 1 mark]

Q.53 The following simultaneous equations
$x + y + z = 3$
$x + 2y + 3z = 4$
$x + 4y + kz = 6$
will NOT have a unique solution for k equal to
(a) 0   (b) 5
(c) 6   (d) 7

[CE, GATE-2008, 2 marks]

Q.54 For what value of a, if any, will the following system of equations in x, y and z have a solution?
$2x + 3y = 4$; $x + y + z = 4$; $x + 2y - z = a$
(a) Any real number   (b) 0
(c) 1   (d) There is no such value

[ME, GATE-2008, 2 marks]

Q.55 The system of linear equations
$4x + 2y = 7$
$2x + y = 6$
has
(a) a unique solution
(b) no solution
(c) an infinite number of solutions
(d) exactly two distinct solutions

[EC, GATE-2008, 1 mark]

Q.56 The following system of equations
$$x_1 + x_2 + 2x_3 = 1$$
$$x_1 + 2x_3 + 3x_3 = 2$$
$$x_1 + 4x_2 + ax_3 = 4$$
has a unique solution. The only possible value(s) for a is/are
(a) 0
(b) either 0 or 1
(c) one of 0,1 or -1
(d) any real number other than 5

[CS, GATE-2008, 1 mark]

Q.57 The Eigen values of the matrix $[P] = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$ are
(a) -7 and 8   (b) -6 and 5
(c) 3 and 4   (d) 1 and 2

[CE, GATE-2008, 2 marks]

Answers	Linear Algebra					

1. (c)	2. (c)	3. (b)	4. (c)	5. (a)	6. (b)	7. (c)
8. (c)	9. (b)	10. (a)	11. (c)	12. (b)	13. (a)	14. (b)
15. (a)	16. (b)	17. (a)	18. (d)	19. (c)	20. (b)	21. (b)
22. (c)	23. (a)	24. (c)	25. (a)	26. (b)	27. (d)	28. (c)
29. (b)	30. (a)	31. (a)	32. (d)	33. (a)	34. (b)	35. (c)
36. (a)	37. (a)	38. (b)	39. (d)	40. (b)	41. (a)	42. (a)
43. (a)	44. (b)	45. (b)	46. (a)	47. (b)	48. (a)	49. (b)
50. (b)	51. (a)	52. (a)	53. (d)	54. (b)	55. (b)	56. (d)
57. (b)	58. (b)	59. (a)	60. (c)	61. (c)	62. (d)	63. (b)
64. (a)	65. (d)	66. (b)	67. (b)	68. (a)	69. (a)	70. (c)
71. (d)	72. (a)	73. (c)	74. (c)	75. (a)	76. (b)	77. (d)
78. (a)	79. (b)	80. (c)	81. (d)	82. (b)	83. (b)	84. (d)
85. (b)	86. (c)	87. (b)	88. (c)	89. (a)	90. (b)	91. (c)
92. (d)	93. (b)	94. (b)	95. (16)	96. (b)	97. (a)	98. (b)
99. (c)	100. (d)	101. (a, d)	102. (c)	103. (d)	104. (a)	105. (c)
106. (c)	107. (a)	108. (a)	109. (23)	110. (a)	111. (d)	112. (a)
113. (d)	114. (b)	115. (88)	116. (a)	117. (200)	118. (49)	119. (0)
120. (2)	121. (c)	122. (b)	123. (b)	124. (1)	125. (a)	126. (d)
127. (d)	128. (3)	129. (1)	130. (0)	131. (6)	132. (a)	133. (a)
134. (0)	135. (c)	136. (a)	137. (b)	138. (4.5)	139. (a)	140. (2)
141. (c)	142. (c)	143. (d)	144. (2)	145. (17)	146. (6)	147. (b)
148. (d)	149. (b)	150. (b)	151. (b)	152. (d)	153. (a)	154. (a)
155. (c)	156. (c)	157. (3)	158. (b)	159. (d)	160. (d)	161. (b)
162. (d)	163. (c)	164. (d)	165. (3)	166. (a)	167. (d)	168. (15)
169. (a)	170. (2)	171. (1)	172. (–6)	173. (0.125)	174. (d)	175. (b)
176. (5)	177. (0)	178. (b)	179. (d)	180. (1)	181. (c)	182. (d)
183. (a)	184. (c)	185. (4)	186. (c)	187. (c)	188. (2)	189. (5)
190. (c)	191. (a)	192. (a)	193. (c)	194. (a)	195. (d)	196. (a)
197. (c)	198. (c)	199. (d)	200. (b)	201. (b)	202. (0.25)	203. (c)
204. (2)	205. (5.5)	206. (1)	207. (25)	208. (a)	209. (c)	210. (3)
211. (d)	212. (b)	213. (c)	214. (b)	215. (3)	216. (d)	217. (3)
218. (c)	219. (10)	220. (b)	221. (c)	222. (d)	223. (d)	224. (b)
225. (d)	226. (c)	227. (12)	228. (b)	229. (d)	230. (c)	

Explanations **Linear Algebra**

1. **(c)**
Consider first 3×3 minors, since maximum possible rank is 3

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and
$$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Since all 3×3 minors are zero, now try 2×2 minors.

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

So, rank = 2

2. **(c)**
Given equation are
$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$
Given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$$

Augmented matrix is $\begin{bmatrix} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{bmatrix}$

By gauss elimination

$$\begin{bmatrix} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r(A) = 2$$
$$r(A|B) = 3$$
Since the rank of coefficient matrix is 2 and rank of argument matrix is 3, which is not equal. Hence system has no solution i.e. system is inconsistent.

3. **(b)**
The augmented matrix for the given system is

$$\begin{bmatrix} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{bmatrix}.$$

Performing Gauss-Elimination on the above matrix

$$\begin{bmatrix} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{bmatrix} \xrightarrow[R_3 - 1/2\,R_1]{R_2 - 2R_1} \begin{bmatrix} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5 - 2\alpha \\ 0 & 3/2 & -6 & 7 - \alpha/2 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3/2\,R_2} \begin{bmatrix} 2 & 1 & -4 & \alpha \\ 0 & 1 & -4 & 5 - 2\alpha \\ 0 & 0 & 0 & \dfrac{5\alpha - 1}{2} \end{bmatrix}$$

Now for infinite solution it is necessary that at least one row must be completely zero.

$$\therefore \quad \frac{5\alpha - 1}{2} = 0$$
$$\alpha = 1/5 \text{ is the solution}$$

∴ There is only one value of α for which infinite solution exists.

4. **(c)**

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Now, $A - \lambda I = 0$
Where $\lambda = $ eigen value

$$\therefore \quad \begin{bmatrix} 4 - \lambda & 1 \\ 1 & 4 - \lambda \end{bmatrix} = 0$$

$$(4 - \lambda)^2 - 1 = 0$$
or, $(4 - \lambda)^2 - (1)^2 = 0$
or, $(4 - \lambda + 1)(4 - \lambda - 1) = 0$
or, $(5 - \lambda)(3 - \lambda) = 0$
$$\therefore \quad \lambda = 3, \lambda = 5$$

5. **(a)**

For singularity of matrix $= \begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix} = 0$

$\Rightarrow 8(0 - 12) - x(0 - 2 \times 12) = 0$
$$\therefore \quad x = 4$$

6. **(b)**

A, B, C, D is $n \times n$ matrix.

Given $\qquad ABCD = I$

$\Rightarrow \quad ABCDD^{-1} C^{-1} = D^{-1} C^{-1}$

$\Rightarrow \qquad\qquad AB = D^{-1} C^{-1}$

$\Rightarrow \qquad A^{-1} AB = A^{-1} D^{-1} C^{-1}$

$\Rightarrow \qquad\qquad B = A^{-1} D^{-1} C^{-1}$

$\qquad\qquad\qquad B^{-1} = (A^{-1} D^{-1} C^{-1})^{-1}$

$\qquad\qquad\qquad\quad = (C^{-1})^{-1} \cdot (D^{-1})^{-1} \cdot (A^{-1})^{-1}$

$\qquad\qquad\qquad\quad = CDA$

7. **(c)**

$\qquad -x + 5y = -1$

$\qquad\quad x - y = 2$

$\qquad\quad x + 3y = 3$

The augmented matrix is $\begin{bmatrix} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$.

Using gauss-elimination on above matrix we get,

$\begin{bmatrix} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 + R_1} \begin{bmatrix} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{bmatrix}$

$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Rank $[A|B] = 2$ (number of non zero rows in $[A|B]$)

$\quad$ Rank $[A] = 2$ (number of non zero rows in $[A]$)

Rank $[A|B] = $ Rank $[A]$

$\qquad\qquad = 2 = $ number of variables

$\therefore$ Unique solution exists. Correct choice is (c).

8. **(c)**

Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 4 - \lambda & -2 \\ -2 & 1 - \lambda \end{vmatrix} = 0$$

$(4 - \lambda) \times (1 - \lambda) - [(-2) \times (-2)] = 0$

$\qquad\qquad \lambda^2 - 5\lambda = 0$

$\Rightarrow \qquad \lambda(\lambda - 5) = 0$

Hence, $\qquad \lambda = 0, 5$ are the eigen values.

9. **(b)**

Sum of eigen values of given matrix = sum of diagonal element of given matrix = $1 + 5 + 1 = 7$.

10. **(a)**

With the given order we can say that order of matrices are as follows:

$X^T \to 3 \times 4$

$Y \to 4 \times 3$

$X^T Y \to 3 \times 3$

$(X^T Y)^{-1} \to 3 \times 3$

$P \to 2 \times 3$

$P^T \to 3 \times 2$

$P(X^T Y)^{-1} P^T \to (2 \times 3)(3 \times 3)(3 \times 2) \to 2 \times 2$

$\therefore (P(X^T Y)^{-1} P^T)^T \to 2 \times 2$

11. **(c)**

For orthogonal matrix

$\qquad AA^T = I \quad$ i.e. Identity matrix.

$\therefore \quad (AA^T)^{-1} = I^{-1} = I$

12. **(b)**

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$R^{-1} = \frac{\text{adj}(R)}{|R|} = \frac{[\text{cofactor }(R)]^T}{|R|}$$

$$|R| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix}$$

$\qquad = 1(2 + 3) - 0(4 + 2) - 1(6 - 2)$

$\qquad = 5 - 4 = 1$

Since we need only the top row of R^{-1}, we need to find only first column of cof (R) which after transpose will become first row of adj (R).

cof. $(1, 1) = + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 + 3 = 5$

cof. $(2, 1) = - \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = -3$

cof. $(3, 1) = + \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = +1$

$\therefore \quad$ cof. $(A) = \begin{bmatrix} 5 & - & - \\ -3 & - & - \\ 1 & - & - \end{bmatrix}$

Adj $(A) = [\text{cof. }(A)]^T = \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$

Dividing by $|R| = 1$ gives

$$R^{-1} = \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

$\therefore \quad$ Top row of $R^{-1} = [5 \quad -3 \quad 1]$

13. (a)

$$[AA^{-1}] = I$$

$$\Rightarrow \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2a - 0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2a - 0.1b = 0 \Rightarrow a = \frac{0.1b}{2} \quad ...(i)$$

$$3b = 1 \Rightarrow b = \frac{1}{3}$$

Now substitute b in equation (i), we get

$$a = \frac{1}{60}$$

So, $\quad a + b = \dfrac{1}{60} + \dfrac{1}{3}$

$$= \frac{1+20}{60} = \frac{21}{60} = \frac{7}{20}$$

14. (b)

$$r(A_{m \times n}) \le \min(m, n)$$

So, Highest possible rank = Least value of 3 and 4. i.e. highest possible rank (based on size of A) = 3 However if the rank of A = 3 then rank of $[A \mid B]$ also would be 3, which means the system would become consistent. But it is given that the system is inconsistent. So the maximum rank of A could only be 2.

15. (a)

Rank $[Pq]$ = Rank $[P]$ is necessary for existence of at least one solution to $Px = q$.

16. (b)

The augmented matrix for the given system is

$$\begin{bmatrix} 2 & -1 & 3 & | & 1 \\ 3 & -2 & 5 & | & 2 \\ -1 & -4 & 1 & | & 3 \end{bmatrix}.$$

Using gauss-elimination method on above matrix we get,

$$\begin{bmatrix} 2 & -1 & 3 & | & 1 \\ 3 & -2 & 5 & | & 2 \\ -1 & -4 & 1 & | & 3 \end{bmatrix} \xrightarrow[\substack{R_3 + \frac{1}{2}R_1}]{R_2 - \frac{3}{2}R_1} \begin{bmatrix} 2 & -1 & 3 & | & 1 \\ 0 & -1/2 & 1/2 & | & 1/2 \\ 0 & -9/2 & 5/2 & | & 7/2 \end{bmatrix}$$

$$\xrightarrow{R_3 - 9R_2} \begin{bmatrix} 2 & -1 & 3 & | & 1 \\ 0 & -1/2 & 1/2 & | & 1/2 \\ 0 & 0 & -2 & | & -1 \end{bmatrix}$$

Rank $([A \mid B])$ = 3

Rank $([A])$ = 3

Since Rank $([A \mid B])$ = Rank $([A])$ = number of variables. The system has unique solution.

17. (a)

First solve for eigen values by solving characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 5 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$= (5-\lambda)(5-\lambda)[(2-\lambda)(1-\lambda)-3]$$
$$= 0$$
$$= (5-\lambda)(5-\lambda)(\lambda^2 - 3\lambda - 1) = 0$$

$$\lambda = 5, 5, \frac{3 \pm \sqrt{13}}{2}$$

put $\lambda = 5$ in $[A - \lambda I]X = 0$

$$\begin{bmatrix} 5-5 & 0 & 0 & 0 \\ 0 & 5-5 & 5 & 0 \\ 0 & 0 & 2-5 & 1 \\ 0 & 0 & 3 & 1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x_3 = 0$; $-3x_3 + x_4 = 0$; $3x_3 - 4x_4 = 0$ Solving which we get $x_3 = 0$, $x_4 = 0$, x_1 and x_2 may be anything.

The eigen vector corresponding to $\lambda = 5$, may be written as

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$$

where k_1, k_2 may be any real number. Since choice (a) is the only matrix in this form with both x_3 and $x_4 = 0$, so it is the correct answer. Since, we already got a correct eigen vector, there is no need to derive the eigen vector corresponding to $\lambda = \dfrac{3 \pm \sqrt{13}}{2}$.

18. (d)

Since matrix is triangular, the eigen values are the diagonal elements themselves namely $\lambda = 3, -2$ and 1. Corresponding to eigen value, $\lambda = -2$ let us find the eigen vector

$$[A - \lambda I]\,x = 0$$

$$\begin{bmatrix} 3-\lambda & -2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = -2$ in above equation we get,

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which gives the equations,

$$5x_1 - 2x_2 + 2x_3 = 0 \qquad \ldots \text{(i)}$$
$$x_3 = 0 \qquad \ldots \text{(ii)}$$
$$3x_3 = 0 \qquad \ldots \text{(iii)}$$

Since eq. (ii) and (iii) are same we have

$$5x_1 - 2x_2 + 2x_3 = 0 \qquad \ldots \text{(i)}$$
$$x_3 = 0 \qquad \ldots \text{(ii)}$$

Putting $x_2 = k$, in eq. (i) we get

$$5x_1 - 2k + 2 \times 0 = 0$$
$$\Rightarrow \qquad x_1 = 2/5\,k$$

∴ Eigen vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/5\,k \\ k \\ 0 \end{bmatrix}$$

i.e. $x_1 : x_2 : x_3 = 2/5\,k : k : 0 = 2/5 : 1 : 0 = 2 : 5 : 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \text{ is an eigen vector of matrix } A.$$

19. (c)

First, find the eigen values of $A = \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-4-\lambda)(3-\lambda) - 8 = 0$$
$$\Rightarrow \lambda^2 + \lambda - 20 = 0$$
$$\Rightarrow (\lambda + 5)(\lambda - 4) = 0$$
$$\Rightarrow \lambda_1 = -5 \text{ and } \lambda_2 = 4$$

Corresponding to $\lambda_1 = -5$ we need to find eigen vector:

The eigen value problem is $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = 0$$

Putting $\lambda = -5$

we get, $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 = 0 \qquad \ldots \text{(i)}$$
$$4x_1 + 8x_2 = 0 \qquad \ldots \text{(ii)}$$

Since (i) and (ii) are the same equation we take

$$x_1 + 2x_2 = 0$$
$$x_1 = -2x_2$$
$$x_1 : x_2 = -2 : 1$$

$$\Rightarrow \qquad \frac{x_1}{x_2} = -2$$

Now from the answers given, we look for any

vector in this ratio and we find choice (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is

in this ratio $\dfrac{x_1}{x_2} = \dfrac{2}{-1} = -2$.

So choice (c) is an eigen vector corresponding to $\lambda = -5$.

Since we already got an answer, there is no need to find the second eigen vector corresponding to $\lambda = 4$.

20. (b)

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

The characteristic equation of this matrix is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$
$$\lambda = 1, 6$$

∴ The eigen values of A are 1 and 6.

21. (b)

Although $\lambda_i^{\,m}$ will be the corresponding eigen values of A^m, $x_i^{\,m}$ need not be corresponding eigen vectors.

22. (c)

Method 1:

$$E = \begin{bmatrix} \cos & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\qquad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

According to problem
$\qquad E \times F = G$

or $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence we see that product of $(E \times F)$ is unit matrix so F has to be the inverse of E.

$$F = E^{-1} = \frac{Adj(E)}{|E|}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 2:

An easier method for finding F is by multiplying E with each of the choices (a), (b), (c) and (d) and finding out which one gives the product as identity matrix G. Again the answer is (c).

23.   (a)

A. Singular matrix $\rightarrow$ Determinant is zero
B. Non-square matrix $\rightarrow$ Determinant is not defined
C. Real symmetric $\rightarrow$ Eigen values are always real
D. Orthogonal matrix $\rightarrow$ Determinant is always one

24.   (c)

Perform, Gauss elimination

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It is in row Echelon form
So its rank is the number of non-zero rows in this form.
i.e., rank = 2

25.   (a)

We are looking for orthogonal vectors having a span that contain P, Q and R.

Take choice (a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

Firstly these are orthogonal, as can be seen by taking their dot product
$= -6 \times 4 + -3 \times -2 + 6 \times 3 = 0$

The space spanned by these two vectors is

$$k_1 \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} + k_2 \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \qquad \dots \text{(i)}$$

The span of $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ contains P, Q and R.

We can show this by successively setting equation (i) to P, Q and R one by one and solving for k_1 and k_2 uniquely.

Notice also that choices (b), (c) and (d) are wrong since none of them are orthogonal as can be seen by taking pairwise dot products.

26.   (b)

The vector $\begin{bmatrix} -2 & -17 & 30 \end{bmatrix}^T$ is linearly dependent upon the solution obtained in previous question namely $\begin{bmatrix} -6 & -3 & 6 \end{bmatrix}^T$ and $\begin{bmatrix} 4 & -2 & 3 \end{bmatrix}^T$.

This can be easily cheeked by finding determinant

of $\begin{bmatrix} -6 & -3 & 6 \\ 4 & -2 & 3 \\ -2 & -17 & 30 \end{bmatrix}$.

$\begin{vmatrix} -6 & -3 & 6 \\ 4 & -2 & 3 \\ -2 & -17 & 30 \end{vmatrix}$
$= -6(-60 + 51) + 3(120 + 6) + 6(-68 - 4) = 0$
Hence, it is linearly dependent.

27.   (d)

The augmented matrix for given system is

$$\begin{bmatrix} 0 & 4 & 3 & | & 8 \\ 2 & 0 & -1 & | & 2 \\ 3 & 2 & 0 & | & 5 \end{bmatrix} \xrightarrow{\text{Exchange 1st and 2nd row}} \begin{bmatrix} 2 & 0 & -1 & | & 2 \\ 0 & 4 & 3 & | & 8 \\ 3 & 2 & 0 & | & 5 \end{bmatrix}$$

then by Gauss elimination procedure

$$\begin{bmatrix} 2 & 0 & -1 & | & 2 \\ 0 & 4 & 3 & | & 8 \\ 3 & 2 & 0 & | & 5 \end{bmatrix} \xrightarrow{R_3-\frac{3}{2}R_1} \begin{bmatrix} 2 & 0 & -1 & | & 2 \\ 0 & 4 & 3 & | & 8 \\ 0 & 2 & 3/2 & | & 2 \end{bmatrix}$$

$$\xrightarrow{R_3-\frac{2}{4}R_2} \begin{bmatrix} 2 & 0 & -1 & | & 8 \\ 0 & 4 & 3 & | & 8 \\ 0 & 0 & 0 & | & -2 \end{bmatrix}$$

For last row we see 0 = –2 which is inconsistent.
Also notice that $r(A) = 2$, while $r(A \mid B) = 3$, $(r(A) \neq r(A \mid B)$ means inconsistent).
∴ Solution is non-existent for above system.

28. **(c)**

$$M = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, [M - \lambda I] = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

Given eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$

$$[M - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} 101 \\ 101 \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda)(101) + 2 \times 101 = 0$$

$$\Rightarrow \qquad\qquad\qquad \lambda = 6$$

29. **(b)**

$$\Sigma\lambda_i = \text{Trace (A)}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace (A)} = 2 + (-1) + 0 = 1$$

Now $\qquad \lambda_1 = 3$

$$\therefore 3 + \lambda_2 + \lambda_3 = 1$$

$$\Rightarrow \qquad \lambda_2 + \lambda_3 = -2$$

Only choice (b) satisfies this condition.

30. **(a)**

If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_4$ are the eigen values of A. Then the eigen values of

A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m \ldots$.

Here, S matrix has eigen values 1 and 5.
So, S^2 matrix has eigen values 1^2 and 5^2 i.e. 1 and 25.

31. **(a)**

By property of eigen values, sum of diagonal elements should be equal to sum of values of λ.
So, $\Sigma \lambda_i = \lambda_1 + \lambda_2 = 8 + 4 = 12 = \text{Trace } (A)$
Only in choice (a), Trace (A) = 12.

32. **(d)**

Since $\qquad S^T = (A + A^T)^T$

$$= A^T + (A^T)^T$$

$$= A^T + A = S$$

i.e. $\qquad S^T = S$

∴ S is symmetric

Since $\qquad D^T = (A - A^T)^T = A^T - (A^T)^T$

$$= A^T - A = -(A - A^T) = -D$$

i.e. $\qquad D^T = -D$

So D is Skew-Symmetric.

33. **(a)**

Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}^{-1} = \frac{1}{(7-10)} \begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{-3} \begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$$

34. **(b)**

If $\qquad X = (x_1, x_2, \ldots x_n)^T$
Rank $X = 1$, since it is non-zero n-tuple.
Rank X^T = Rank $X = 1$
Now Rank $(X^T) \leq \min(\text{Rank } X, \text{Rank } X^T)$

$$\Rightarrow \quad \text{Rank } (XX^T) \leq \min(1, 1)$$

$$\Rightarrow \quad \text{Rank } (XX^T) \leq 1.$$

So XX^T has a rank of a either 0 or 1.
But since both X and X^T are non-zero vectors, so neither of their ranks can be zero.
So XX^T has a rank 1.

35. **(c)**

Since $(X_1, X_2, \ldots X_M)$ are orthogonal, they span a vector space of dimension M.
Since $(-X_1, -X_2, \ldots X_M)$ are linearly dependent on $X_1, X_2, \ldots X_M$, the set $(X_1, X_2, X_3, \ldots X_M, -X_1, -X_2, \ldots -X_M)$ will also span a vector space of dimension M only.

36. **(a)**

To be basis for subspace X, two conditions are to be satisfied
1. The vectors have to be linearly independent.
2. They must span X.

Here, $\qquad X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$

$$x^T = [x_1, x_2, x_3]^T$$

Step 1: Now, $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set because one cannot be obtained from another by scalar multiplication. The fact that it is independent can also be established by

seeing that rank of $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ is 2.

Step 2: Next, we need to check if the set spans X.

Here, $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$

The general infinite solution of $X = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$

Choosing k_1, k_2 as $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix}$ and $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$,

we get 2 linearly independent solutions, for X,

$$X = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} \text{ or } \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix}$$

Now since both of these can be generated by linear combinations of $[1, -1, 0]^T$ and $[1, 0, -1]^T$, the set spans X. Since we have shown that the set is not only linearly independent but also spans X, therefore by definition it is a basis for the subspace X.

37. (a)

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 1 & | & 5 \\ 1 & 3 & 3 & | & 9 \\ 1 & 2 & \alpha & | & \beta \end{bmatrix}$$

Using Gauss-elimination method we get

$$\begin{bmatrix} 1 & 1 & 1 & | & 5 \\ 1 & 3 & 3 & | & 9 \\ 1 & 2 & \alpha & | & \beta \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & | & 5 \\ 0 & 2 & 2 & | & 4 \\ 0 & 1 & \alpha - 1 & | & \beta - 5 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & | & 5 \\ 0 & 2 & 2 & | & 4 \\ 0 & 0 & \alpha - 2 & | & \beta - 7 \end{bmatrix}$$

Now, for infinite solution last row must be completely zero
i.e. $\alpha - 2$ and $\beta - 7 = 0$
$\Rightarrow \qquad \alpha = 2$ and $\beta = 7$

38. (b)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$[a - \lambda I] = 0$$

$$\begin{bmatrix} 2 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = 0$$

$\Rightarrow \qquad (2 - \lambda)^2 = 0$
$\Rightarrow \qquad \lambda = 2$

Now, consider the eigen value problem

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 2 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

put $\lambda = 2$, we get,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0 \qquad \qquad \text{... (i)}$$
$$0 = 0 \qquad \qquad \text{... (ii)}$$

The solution is therefore $x_2 = 0$, $x_1 = $ anything

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

39. (d)

The cross product of b = $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$

and $X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ can be written as

$$b \times X = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

$$= x_3 \hat{i} + 0 \hat{j} - x_1 \hat{k}$$

$$= \begin{bmatrix} x_3 & 0 & -x_1 \end{bmatrix}$$

Now $\qquad L(x) = b \times X = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

where M is a 3×3 matrix

Let $\qquad M = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$

Now $M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b \times X$

$$\Rightarrow \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

By matching LHS and RHS we get

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

So, $\qquad M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Now we have to find the eigen values of M

$$|M - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 0) + 1(0 - \lambda) = 0$$

$$\Rightarrow \lambda^3 + \lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 + 1) = 0$$

$$\Rightarrow \lambda = 0, \lambda = \pm i$$

So, the eigen values of M are i, $-i$ and 0.

40. (b)

$$\Sigma\lambda_i = \text{Trace (A)}$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1 = 7$$
Now, $\lambda_1 = -2, \lambda_2 = 6$
$$\therefore \quad -2 + 6 + \lambda_3 = 7$$
$$\lambda_3 = 3$$

41. (a)

The eigen values of any symmetric matrix is always real.

42. (a)

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -3 - \lambda & 2 \\ -1 & 0 - \lambda \end{vmatrix} = 0$$

$$(-3 - \lambda)(-\lambda) + 2 = 0$$
$$\lambda^2 + 3\lambda + 2 = 0$$

A will satisfy this equation according to Caylay-Hamilton theorem
i.e. $A^2 + 3A + 2I = 0$
multiplying by A^{-1} on both sides we get
$$A^{-1} A^2 + 3A^{-1} A + 2A^{-1} I = 0$$
$$A + 3I + 2A^{-1} = 0$$

43. (a)

To calculate A^9
start from $A^2 + 3A + 2I = 0$ which has been derived above

$$\Rightarrow \quad A^2 = -3A - 2I$$
$$A^4 = A^2 \times A^2 = (-3A - 2I)(-3A - 2I)$$
$$= 9A^2 + 12A + 4I$$
$$= 9(-3A - 2I) + 12A + 4I$$
$$= -15A - 14I$$
$$A^8 = A^4 \times A^4$$
$$= (-15A - 14I)(-15A - 14I)$$

$$= 225A^2 + 420A + 156I$$
$$= 225(-3A - 2I) + 420A\ 196I$$
$$= -255A - 254I$$
$$A^9 = A \times A^8$$
$$= A(-255A - 254I)$$
$$= -255A^2 - 254A$$
$$= -255(-3A - 2I) - 254A$$
$$= 511A + 510I$$

44. (b)

Given, $A^2 = I$

$$|A^2| = |I|$$
$$|A| \cdot |A| = 1$$
$$|A| = \pm 1$$

So, $|A| \neq 0$, so system of equations $AX = Y$ is consistent, and has unique solution given by $X = A^{-1} Y$.

45. (b)

$$A = \begin{bmatrix} 1\cdot1 & 1\cdot2 & 1\cdot3 & \dots & 1\cdot n \\ 2\cdot1 & 2\cdot2 & 2\cdot3 & \dots & 2\cdot n \\ 3\cdot1 & 3\cdot2 & 3\cdot3 & \dots & 3\cdot n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n\cdot1 & n\cdot2 & n\cdot3 & \dots & n\cdot n \end{bmatrix}$$

All row are the multiple of first row

$$\therefore \quad A = \begin{bmatrix} 1\cdot1 & 1\cdot2 & 1\cdot3 & \dots & 1\cdot n \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Rank of $A = 1$.

46. (a)

$$\begin{vmatrix} 1 + b & b & 1 \\ b & 1 + b & 1 \\ 1 & 2b & 1 \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$\begin{vmatrix} 2(1 + b) & b & 1 \\ 2(1 + b) & 1 + b & 1 \\ 2(1 + b) & 2b & 1 \end{vmatrix}$$

$$= 2(1 + b) \begin{vmatrix} 1 & b & 1 \\ 1 & 1 + b & 1 \\ 1 & 2b & 1 \end{vmatrix}$$

$$= 0 \text{ [Since } C_1 \text{ and } C_3 \text{ are identical)]}$$

47. (b)

$A_{2n \times 2n}$ matrix has '2n' real eigen values which may/may not be distinct.

48. (a)

$A = IA$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}[A]$$

$$R_1 \longleftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}[A]$$

$$\Rightarrow I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}[A]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

49. (b)

Characteristic equation,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} (3-\lambda) & 4 \\ 4 & -(3+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow -(3-\lambda)(3+\lambda) - 16 = 0$$
$$\Rightarrow 9 - \lambda^2 + 16 = 0$$
$$\Rightarrow \lambda^2 = 25$$
$$\Rightarrow \lambda = \pm 5$$

For $\lambda = +5$

$$(A - \lambda I) X = 0$$

$$\Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 + 2R_1$$

$$\Rightarrow \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad ...(1)$$

$\Rightarrow \rho(A - \lambda I) = 1$: Number of variables = 2

$\Rightarrow$ Free variables (Nullity) = 1

Let $\qquad x_1 = K \qquad ...(2)$

$\Rightarrow \qquad -2K + 4x_2 = 0$ (from (1) and (2))

$\Rightarrow \qquad x_2 = (1/2)K$

Eigen vector : $\begin{bmatrix} K \\ 0.5K \end{bmatrix}$ for $K = 2$;

Eigen vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Also, $AA^{-1} = I$

from option. **Ans.** $\rightarrow$ (b)

Check: Eigen vectors corresponding to different eigen values of a symmetric matrix are orthogonal.

$$\Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} = 2 - 2 = 0 \Rightarrow \text{Orthogonal}$$

Alter.: Determine eigen vector for $\lambda = -5$ using

$$(A - \lambda I) X = 0$$

50. (b)

$$(PQ)^{-1} P = (Q^{-1} P^{-1})P$$
$$= (Q^{-1})(P^{-1}P) = (Q^{-1})(I)$$
$$= Q^{-1}$$

51. (a)

Choice (a) $AA'A = A$ is correct

Since, $\qquad AA'A = A[(A^TA)^{-1} A^T]A$
$$= A[(A^TA)^{-1} A^TA]$$

Let, $\qquad A^TA = P$

Then, $\qquad = A[P^{-1} P] = A \cdot I = A$

52. (a)

If rank of (5×6) matrix is 4, then surely it must have exactly 4 linearly independent rows as well as 4 linearly independent columns, since rank = row rank = column rank.

53. (d)

The augmented matrix for given system is

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 1 & 2 & 3 & | & 4 \\ 1 & 4 & k & | & 6 \end{bmatrix}.$$

Using Gauss elimination we reduce this to an upper triangular matrix to investigate its rank.

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 1 & 2 & 3 & | & 4 \\ 1 & 4 & k & | & 6 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 2 & | & 1 \\ 0 & 3 & k-1 & | & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & k-7 & | & 0 \end{bmatrix}$$

Now if $\qquad k \neq 7$

$\qquad$ rank (A) = rank $(A | B)$ = 3

$\therefore$ unique solution

If $\qquad k = 7$, rank (A) = rank $(A | B)$ = 2

which is less than number of variables

$\therefore$ when $k = 7$, unique solution is not possible and only infinite solution is possible.