Thoroughly Revised and Updated

Engineering Mathematics

For

GATE 2017 and ESE 2017 Prelims

Note: ESE Mains Electrical Engineering also covered

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Engineering Mathematics for GATE and ESE Prelims - 2017: Illusatrative Examples, Topicwise Previous GATE Solved Papers (2003-2016)

First Edition	:	2009
Second Edition	:	2010
Third Edition	:	2011
Fourth Edition	:	2012
Fifth Edition	:	2013
Sixth Edition	:	2014
Seventh Edition	:	2015
Eight Edition	**:**	**2016**

Preface

B. Singh (Ex. IES)

Over the period of time the GATE and ESE examination have become more challenging due to increasing number of candidates. Though every candidate has ability to succeed but competitive environment, in-depth knowledge, quality guidance and good source of study is required to achieve high level goals.

The new edition of **Engineering Mathematics for GATE and ESE Prelims - 2017** has been fully revised, updated and edited. The whole book has been divided into topicwise sections.

I have true desire to serve student community by way of providing good source of study and quality guidance. I hope this book will be proved an important tool to succeed in GATE and ESE examination. Any suggestions from the readers for the improvement of this book are most welcome.

B. Singh (Ex. IES)
Chairman and Managing Director
MADE EASY Group

SYLLABUS

GATE and ESE Prelims: Civil Engineering

Linear Algebra: Matrix algebra; Systems of linear equations; Eigen values and Eigen vectors.

Calculus: Functions of single variable; Limit, continuity and differentiability; Mean value theorems, local maxima and minima, Taylor and Maclaurin series; Evaluation of definite and indefinite integrals, application of definite integral to obtain area and volume; Partial derivatives; Total derivative; Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Ordinary Differential Equation (ODE): First order (linear and non-linear) equations; higher order linear equations with constant coefficients; Euler-Cauchy equations; Laplace transform and its application in solving linear ODEs; initial and boundary value problems.

Partial Differential Equation (PDE): Fourier series; separation of variables; solutions of one-dimensional diffusion equation; first and second order one-dimensional wave equation and two-dimensional Laplace equation.

Probability and Statistics: Definitions of probability and sampling theorems; Conditional probability; Discrete Random variables: Poisson and Binomial distributions; Continuous random variables: normal and exponential distributions; Descriptive statistics - Mean, median, mode and standard deviation; Hypothesis testing.

Numerical Methods: Accuracy and precision; error analysis. Numerical solutions of linear and non-linear algebraic equations; Least square approximation, Newton's and Lagrange polynomials, numerical differentiation, Integration by trapezoidal and Simpson's rule, single and multi-step methods for first order differential equations.

GATE and ESE Prelims: Mechanical Engineering

Linear Algebra: Matrix algebra, systems of linear equations, eigenvalues and eigenvectors.

Calculus: Functions of single variable, limit, continuity and differentiability, mean value theorems, indeterminate forms; evaluation of definite and improper integrals; double and triple integrals; partial derivatives, total derivative, Taylor series (in one and two variables), maxima and minima, Fourier series; gradient, divergence and curl, vector identities, directional derivatives, line, surface and volume integrals, applications of Gauss, Stokes and Green's theorems.

Differential equations: First order equations (linear and nonlinear); higher order linear differential equations with constant coefficients; Euler-Cauchy equation; initial and boundary value problems; Laplace transforms; solutions of heat, wave and Laplace's equations.

Complex Variables: Analytic functions; Cauchy-Riemann equations; Cauchy's integral theorem and integral formula; Taylor and Laurent series.

Probability and Statistics: Definitions of probability, sampling theorems, conditional probability; mean, median, mode and standard deviation; random variables, binomial, Poisson and normal distributions.

Numerical Methods: Numerical solutions of linear and non-linear algebraic equations; integration by trapezoidal and Simpson's rules; single and multi-step

methods for differential equations..

GATE and ESE Prelims: Electrical Engineering

Linear Algebra: Matrix Algebra, Systems of linear equations, Eigenvalues, Eigenvectors.

Calculus: Mean value theorems, Theorems of integral calculus, Evaluation of definite and improper integrals, Partial Derivatives, Maxima and minima, Multiple integrals, Fourier series, Vector identities, Directional derivatives, Line integral, Surface integral, Volume integral, Stokes's theorem, Gauss's theorem, Green's theorem.

Differential equations: First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Method of variation of parameters, Cauchy's equation, Euler's equation, Initial and boundary value problems, Partial Differential Equations, Method of separation of variables.

Complex Variables: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula, Taylor series, Laurent series, Residue theorem, Solution integrals.

Probability and Statistics: Sampling theorems, Conditional probability, Mean, Median, Mode, Standard Deviation, Random variables, Discrete and Continuous distributions, Poisson distribution, Normal distribution, Binomial distribution, Correlation analysis, Regression analysis.

Numerical Methods: Solutions of nonlinear algebraic equations, Single and Multi-step methods for differential equations.

Transform Theory: Fourier Transform, Laplace Transform, z-Transform.

Electrical Engineering ESE Mains

Matrix theory, Eigen values & Eigen vectors, system of linear equations, Numerical methods for solution of non-linear algebraic equations and differential equations, integral calculus, partial derivatives, maxima and minima, Line, Surface and Volume Integrals. Fourier series, linear, nonlinear and partial differential equations, initial and boundary value problems, complex variables, Taylor's and Laurent's series, residue theorem, probability and statistics fundamentals, Sampling theorem, random variables, Normal and Poisson distributions, correlation and regression analysis.

GATE and ESE Prelims: Electronics Engineering

Linear Algebra: Vector space, basis, linear dependence and independence, matrix algebra, eigen values and eigen vectors, rank, solution of linear equations — existence and uniqueness.

Calculus: Mean value theorems, theorems of integral calculus, evaluation of definite and improper integrals, partial derivatives, maxima and minima, multiple integrals, line, surface and volume integrals, Taylor series.

Differential equations: First order equations (linear and nonlinear), higher order linear differential equations, Cauchy's and Euler's equations, methods of solution using variation of parameters, complementary function and particular integral, partial differential equations, variable separable method, initial and boundary value problems.

Vector Analysis: Vectors in plane and space, vector operations, gradient, divergence and curl, Gauss's, Green's and Stoke's theorems.

Complex Analysis: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula; Taylor's and Laurent's series, residue theorem.

Numerical Methods: Solution of nonlinear equations, single and multi-step methods for differential equations, convergence criteria.

Probability and Statistics: Mean, median, mode and standard deviation; combinatorial probability, probability distribution functions - binomial, Poisson, exponential and normal; Joint and conditional probability; Correlation and regression analysis.

GATE: Instrumentation Engineering

Linear Algebra : Matrix algebra, systems of linear equations, Eigen values and Eigen vectors.

Calculus : Mean value theorems, theorems of integral calculus, partial derivatives, maxima and minima, multiple integrals, Fourier series, vector identities, line, surface and volume integrals, Stokes, Gauss and Green's theorems.

Differential Equations : First order equation (linear and nonlinear), higher order linear differential equations with constant coefficients, method of variation of parameters, Cauchy's and Euler's equations, initial and boundary value problems, solution of partial differential equations: variable separable method.

Analysis of complex variables: : Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent's series, residue theorem, solution of integrals.

Complex Variables : Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent' series, Residue theorem, solution integrals.

Probability and Statistics : Sampling theorems, conditional probability, mean, median, mode and standard deviation, random variables, discrete and continuous distributions: normal, Poisson and binomial distributions.

Numerical Methods : Matrix inversion, solutions of non-linear algebraic equations, iterative methods for solving differential equations, numerical integration, regression and correlation analysis.

GATE: Computer Science & IT Engineering

Linear Algebra: Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LU decomposition.

Calculus: Limits, continuity and differentiability. Maxima and minima. Mean value theorem. Integration.

Probability: Random variables. Uniform, normal, exponential, poisson and binomial distributions. Mean, median, mode and standard deviation. Conditional probability and Bayes theorem.

Contents

CHAPTER 1

Linear Algebra

1.1 INTRODUCTION

Linear Algebra is a branch of mathematics concerned with the study of vectors, with families of vectors called vector spaces or linear spaces and with functions that input one vector and output another, according to certain rules. These functions are called linear maps or linear transformations and are often represented by matrices. Matrices are rectangular arrays of numbers or symbols and matrix algebra or linear algebra provides the rules defining the operations that can be formed on such an object.

Linear Algebra and matrix theory occupy an important place in modern mathematics and has applications in almost all branches of engineering and physical sciences. An elementary application of linear algebra is to the solution of a system of linear equations in several unknowns, which often result when linear mathematical models are constructed to represent physical problems. Nonlinear models can often be approximated by linear ones. Other applications can be found in computer graphics and in numerical methods.

In this chapter, we shall discuss matrix algebra and its use in solving linear system of algebraic equations $A\hat{x} = b$ and in solving the eigen value problem $A\hat{x} = \lambda\hat{x}$.

1.2 ALGEBRA OF MATRICES

1.2.1 Definition of Matrix

A system of mn numbers arranged in the form of a rectangular array having m rows and n columns is called an matrix of order $m \times n$.

If $A = [a_{ij}]_{m \times n}$ be any matrix of order $m \times n$ then it is written in the form:

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots\ldots a_{2n} \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix}$$

Horizontal lines are called rows and vertical lines are called columns.

1.2.2 Special Types of Matrices

1. **Square Matrix:** An $m \times n$ matrix for which $m = n$ (The number of rows is equal to number of columns) is called square matrix. It is also called an n-rowed square matrix. i.e. The elements $a_{ij} \mid i = j$, i.e. a_{11}, a_{22}.... are called **DIAGONAL ELEMENTS** and the line along which they lie is called **PRINCIPLE DIAGONAL** of matrix. Elements other than a_{11}, a_{22}, etc are called off-diagonal elements i.e. $a_{ij} \mid i \neq j$.

   Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 3 \end{bmatrix}_{3 \times 3}$ is a square Matrix

   Note: A square sub-matrix of a square matrix A is called a **"principle sub-matrix"** if its diagonal elements are also the diagonal elements of the matrix A. So $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$ is a principle sub matrix of the matrix A given above, but $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ is not.

2. **Diagonal Matrix:** A square matrix in which all off-diagonal elements are zero is called a diagonal matrix. The diagonal elements may or may not be zero.

   Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is a diagonal matrix

   The above matrix can also be written as A = diag [3, 5, 9]

   Properties of Diagonal Matrix:
   diag [x, y, z] + diag [p, q, r] = diag [x + p, y + q, z + r]
   diag [x, y, z] $\times$ diag [p, q, r] = diag [xp, yq, zr]
   $(\text{diag}\,[x, y, z])^{-1} = \text{diag}\,[1/x, 1/y, 1/z]$
   $(\text{diag}\,[x, y, z])^{t} = \text{diag}\,[x, y, z]$
   $(\text{diag}\,[x, y, z])^{n} = \text{diag}\,[x^n, y^n, z^n]$
   Eigen values of diag [x, y, z] = x, y and z.
   Determinant of diag [x, y, z] = $|\text{diag}\,[x, y, z]| = xyz$

3. **Scalar Matrix:** A scalar matrix is a diagonal matrix with all diagonal elements being equal.

   Example: $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix.

4. **Unit Matrix or Identity Matrix:** A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I. Identity matrix is always square.
   Thus a square matrix $A=[a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.

   Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unit matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Properties of Identity Matrix:

(a) I is Identity element for multiplication, so it is called multiplicative identity.

(b) AI = IA = A

(c) $I^n = I$

(d) $I^{-1} = I$

(e) $|I| = 1$

5. **Null Matrix:** The m × n matrix whose elements are all zero is called null matrix.
   Null matrix is denoted by O. Null matrix need not be square.

Example: $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Properties of Null Matrix:

(a) A + O = O + A = A
   So, O is additive identity.

(b) A + (–A) = O

6. **Upper Triangular Matrix:** An upper triangular matrix is a square matrix whose lower off-diagonal elements are zero, i.e. $a_{ij} = 0$ whenever $i > j$
   It is denoted by U.
   The diagonal and upper off diagonal elements may or may not be zero.

Example: $U = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 5 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

7. **Lower Triangular Matrix:** A lower triangular matrix is a square matrix whose upper off-diagonal triangular elements are zero, i.e. $a_{ij} = 0$ whenever $i < j$. The diagonal and lower off-diagonal elements may or may not be zero.
   It is denoted by L,

Example: $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 5 & 0 \\ 2 & 3 & 6 \end{bmatrix}$

8. **Idempotent Matrix:** A matrix A is called Idempotent iff $A^2 = A$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are examples of Idempotent matrices.

9. **Involutory Matrix:** A matrix A is called Involutory iff $A^2 = I$.

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is Involutory. Also $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is Involutory since $A^2 = I$.

10. **Nilpotent Matrix:** A matrix A is said to be nilpotent of class x or index x iff $A^x = O$ and $A^{x-1} \neq O$ i.e. x is the smallest index which makes $A^x = O$.

Example: The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since $A \neq 0$ and $A^2 \neq 0$, but $A^3 = 0$.

1.2.3 Equality of Two Matrices

Two matrices $A= [a_{ij}]$ and $B=[b_{ij}]$ are said to be equal if,

1. They are of same size.
2. The elements in the corresponding places of two matrices are the same i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j.

Example: Let $\begin{bmatrix} x-y & p+q \\ p-q & x+y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 10 \end{bmatrix}$

Then $x - y = 2$, $p + q = 5$, $p - q = 1$ and $x + y = 10$

$\Rightarrow \quad x = 6, y = 4, p = 3$ and $q = 2$.

1.2.4 Addition of Matrices

Two matrices A and B are compatible for addition only if they both have exactly the same size say $m \times n$. Then their sum is defined to be the matrix of the type $m \times n$ obtained by adding corresponding elements of A and B. Thus if, $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{m \times n}$ then $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}$;

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 13 \end{bmatrix}$$

Properties of Matrix Addition:

1. Matrix addition is commutative $A + B = B + A$.
2. Matrix addition is associative $(A + B) + C = A + (B + C)$
3. Existence of additive identity: If O be $m \times n$ matrix each of whose elements are zero. Then, $A + O = A = O + A$ for every $m \times n$ matrix A.
4. Existence of additive inverse: Let $A =[a_{ij}]_{m \times n}$.
   Then the negative of matrix A is defined as matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$.
   $\Rightarrow$ Matrix $-A$ is additive inverse of A. Because $(-A) + A = O = A + (-A)$. Here O is null matrix of order $m \times n$.
5. Cancellation laws holds good in case of addition of matrices, which is $X = -A$.
   $A+X = B +X \Rightarrow A = B$
   $X + A = X + B \Rightarrow A = B$
6. The equation $A + X = 0$ has a unique solution in the set of all $m \times n$ matrices.

1.2.5 Substraction of Two Matrices

If A and B are two $m \times n$ matrices, then we define, $A - B = A + (-B)$.

Thus the difference $A - B$ is obtained by subtracting from each element of A corresponding elements of B.

Note: Subtraction of matrices is neither commutative nor associative.

1.2.6 Multiplication of a Matrix by a Scalar

Let A be any m × n matrix and k be any real number called scalar. The m × n matrix obtained by multiplying every element of the matrix A by k is called scalar multiple of A by k and is denoted by kA.

$\Rightarrow$ If $A = [a_{ij}]_{m \times n}$ then $Ak = kA = [kA]_{m \times n}$.

If $A = \begin{bmatrix} 5 & 2 & 1 \\ 6 & -5 & 2 \\ 1 & 3 & 6 \end{bmatrix}$ then, $3A = \begin{bmatrix} 15 & 6 & 3 \\ 18 & -15 & 6 \\ 3 & 9 & 18 \end{bmatrix}$

Properties of Multiplication of a Matrix by a Scalar:

1. Scalar multiplication of matrices distributes over the addition of matrices i.e., $k(A+B) = kA + kB$.
2. If p and q are two scalars and A is any m × n matrix then, $(p + q)A = pA + qA$.
3. If p and q are two matrices and $A = [a_{ij}]_{m \times n}$ then, $p(qA) = (pq)A$.
4. If $A=[a_{ij}]_{m \times n}$ be a matrix and k be any scalar then, $(-k)A = -(kA) = k(-A)$.

1.2.7 Multiplication of Two Matrices

Let $A = [a_{ij}]_{m \times n}$; $B=[b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the matrix $C = [c_{ik}]_{m \times p}$ such that $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ is called the product of matrices A and B in that order and we write C = AB.

Properties of Matrix Multiplication:

1. Multiplication of matrices is not commutative. In fact, if the product of AB exists, then it is not necessary that the product of BA will also exist. For example, $A_{3 \times 2} \times B_{2 \times 4} = C_{3 \times 4}$ but $B_{2 \times 4} \times A_{3 \times 2}$ does not exist since these are not compatible for multiplication.
2. Matrix multiplication is associative, if conformability is assured. i.e., A (BC) = (AB)C where A, B, C are m × n, n × p, p × q matrices respectively.
3. Multiplication of matrices is distributive with respect to addition of matrices. i.e., A (B+C) = AB + AC.
4. The equation AB = O does not necessarily imply that at least one of matrices A and B must be a zero matrix. For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
5. In the case of matrix multiplication if AB = O then it is not necessarily imply that BA = O. In fact, BA may not even exist.
6. Both left and right cancellation laws hold for matrix multiplication as shown below:
   AB = AC $\Rightarrow$ B = C (iff A is non-singular matrix) and
   BA = CA $\Rightarrow$ B = C (iff A is non-singular matrix).

ILLUSTRATIVE EXAMPLES FROM GATE

Q.1 Consider the matrices $X_{(4 \times 3)}$, $Y_{(4 \times 3)}$ and $P_{(2 \times 3)}$. The order of $[P(X^T Y)^{-1} P^T]^T$ will be

(a) (2×2)
(b) (3×3)
(c) (4×3)
(d) (3×4)

[CE, GATE-2005, 1 mark]

Solution: (a)

With the given order we can say that order of matrices are as follows:

$$\begin{aligned} X^T &\rightarrow 3\times 4 \\ Y &\rightarrow 4\times 3 \\ X^TY &\rightarrow 3\times 3 \\ (X^TY)^{-1} &\rightarrow 3\times 3 \\ P &\rightarrow 2\times 3 \\ P^T &\rightarrow 3\times 2 \\ P(X^TY)^{-1}P^T &\rightarrow (2\times 3)\,(3\times 3)\,(3\times 2) \rightarrow 2\times 2 \\ \therefore \quad (P(X^TY)^{-1}P^T)^T &\rightarrow 2\times 2 \end{aligned}$$

Q.2 There are three matrixes $P(4\times 2)$, $Q(2\times 4)$ and $R(4\times 1)$. The minimum of multiplication required to compute the matrix PQR is

[CE, GATE-2013, 1 Mark]

Solution:

The minimum number of multiplications required to multiply $A_{m\times n}$ with $B_{n\times p}$ is mnp. To compute PQR if we multiply PQ first and then R the number of multiplications required would be $4\times 2\times 4$ to get PQ and then $4\times 4\times 1$ multiplications to mulitply PQ with R. So total multiplications required in this method is

$$4\times 2\times 4 + 4\times 4\times 1 = 32 + 16 = 48$$

To compute PQR if we multiply QR first and then P the number of multiplications required would be $2\times 4\times 1$ to get QR and then $4\times 2\times 1$ multiplications to mulitply P with QR. So total multiplications required in this method is

$$2\times 4\times 1 + 4\times 2\times 1 = 8 + 8 = 16$$

Therfore, the minimum of multiplication required to compute the matrix PQR is = 16

Q.3 Multiplication of matrices E and F is G. Matrices E and G are

$$E \equiv \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What is the matrix F?

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[ME, GATE-2006, 2 marks]

Solution: (c)

Method 1:

$$E = \begin{bmatrix} \cos & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

According to problem

$$E \times F = G$$

or $$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we see that product of $(E \times F)$ is unit matrix so F has to be the inverse of E.

$$F = E^{-1} = \frac{\text{Adj}(E)}{|E|} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 2:

An easier method for finding F is by multiplying E with each of the choices (a), (b), (c) and (d) and finding out which one gives the product as identity matrix G. Again the answer is (c).

1.2.8 Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements lying along principal diagonal is called the trace of A denoted by Tr(A).

Thus if $A = [a_{ij}]_{n \times n}$ then, $Tr(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \ldots a_{nn}$.

Let $$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 6 & 5 \end{bmatrix}$$

Then, trace (A) = tr(A) = 1+(–3) + 5 = 3

Properties of Trace of a Matrix:

Let A and B be two square matrices of order n and λ be a scalar. Then,

1. $tr(\lambda A) = \lambda \, tr\, A$
2. $tr(A + B) = tr\, A + tr\, B$
3. $tr(AB) = tr(BA)$

1.2.9 Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$. Then the $n \times m$ matrix obtained from A by changing its rows into columns and its columns into rows is called the transpose of A and is denoted by A' or A^T.

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$ then, $A^T = A' = \begin{bmatrix} 1 & 2 & 6 \\ 3 & 4 & 5 \end{bmatrix}$

If B = [1 2 3] then

$$B' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Properties of Transpose of a Matrix:

If A' and B' be transposes of A and B respectively then,

1. $(A')' = A$
2. $(A + B)' = A' + B'$
3. $(kA)' = kA'$, k being any complex number
4. $(AB)' = B'A'$
5. $(ABC)' = C'B'A'$

1.2.10 Conjugate of a Matrix

The matrix obtained from given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$.

Example: If $A = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$

$$\overline{A} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ +i & 6 & 9-i \end{bmatrix}$$

Properties of Conjugate of a Matrix:

If $\overline{A}$ & $\overline{B}$ be the conjugates of A & B respectively. Then,

1. $\overline{(\overline{A})} = A$
2. $\overline{(A+B)} = \overline{A}+\overline{B}$
3. $\overline{(kA)} = \overline{k}\,\overline{A}$, k being any complex number
4. $\overline{(AB)} = \overline{A}\,\overline{B}$, A & B being conformable to multiplication
5. $\overline{A} = A$ iff A is real matrix
   $\overline{A} = -A$ iff A is purely imaginary matrix

1.2.11 Transposed Conjugate of Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ or A^* or $\left(\overline{A}\right)^T$. It is also called conjugate transpose of A.

Example: If $A = \begin{bmatrix} 2+i & 3-i \\ 4 & 1-i \end{bmatrix}$

To find A^θ, we first find $\overline{A} = \begin{bmatrix} 2-i & 3+i \\ 4 & 1+i \end{bmatrix}$

Then $A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} 2-i & 4 \\ 3+i & 1+i \end{bmatrix}$

Some properties: If A^θ & B^θ be the transposed conjugates of A and B respectively then,

1. $(A^\theta)^\theta = A$
2. $(A + B)^\theta = A^\theta + B^\theta$
3. $(kA)^\theta = \overline{k}A^\theta$, $k \to$ complex number
4. $(AB)^\theta = B^\theta A^\theta$

1.2.12 Classification of Real Matrices

Real matrices can be classified into the following three types based on the relationship between A^T and A.

1. Symmetric Matrices ($A^T = A$)
2. Skew Symmetric Matrices ($A^T = -A$)
3. Orthogonal Matrices ($A^T = A^{-1}$ or $AA^T = I$)

1. **Symmetric Matrix:** A square matrix A= $[a_{ij}]$ is said to be symmetric if its $(i, j)^{th}$ elements is same as its $(j, i)^{th}$ element i.e., $a_{ij} = a_{ji}$ for all i & j.
   In a symmetric matrix, $A^T = A$

   Example: $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix, since $A^T = A$.

   Note: For any matrix A,
   (a) AA^t is always a symmetric matrix.
   (b) $\dfrac{A + A^t}{2}$ is always symmetric matrix.

   Note: If A and B an symmetric, then
   (a) A + B and A – B are also symmetric.
   (b) AB, BA may or may not be symmetric.

2. **Skew Symmetric Matrix:** A square matrix A = $[a_{ij}]$ is said to be skew symmetric if $(i, j)^{th}$ elements of A is the negative of the $(j, i)^{th}$ elements of A if $a_{ij} = -a_{ji}\ \forall$ i, j.
   In a skew symmetric matrix $A^T = -A$.
   A skew symmetric matrix must have all 0's in the diagonal.

   Example: $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

   Note: For any matrix A, the matrix $\dfrac{A - A^t}{2}$ is always skew symmetric.

3. **Orthogonal Matrix:** A square matrix A is said be orthogonal if:
   $A^T = A^{-1} \Rightarrow AA^T = AA^{-1} = I$. Thus A will be an orthogonal matrix if, $AA^T = I = A^TA$.
   Example: The identity matrix is orthogonal since $I^t = I^{-1} = I$.
   Note: Since for an orthogonal matrix A,

$$
\begin{aligned}
AA^T &= I \\
\Rightarrow \quad |AA^T| &= |I| = 1 \\
\Rightarrow \quad |A|\,|A^T| &= 1 \\
\Rightarrow \quad (|A|)^2 &= 1 \\
\Rightarrow \quad |A| &= \pm 1
\end{aligned}
$$

   So the determinant of an orthogonal matrix always has a modulus of 1.

1.2.13 Classification of Complex Matrices

Complex matrices can be classified into the following three types based on relationship between A^θ and A.

1. Hermitian Matrix ($A^\theta = A$)
2. Skew-Hermitian Matrix ($A^\theta = -A$)
3. Unitary Matrix ($A^\theta = A^{-1}$ or $AA^\theta = I$)

1. **Hermitian Matrix:** A necessary and sufficient condition for a matrix A to be Hermitian is that $A^\theta = A$.

   Example: $A = \begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$ is a Hermitian matrix.

2. **Skew-Hermitian Matrix:** A necessary and sufficient condition for a matrix to be skew-Hermitian if $A^\theta = -A$.

   Example: $A = \begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}$ is skew-Hermitian.

3. **Unitary Matrix:** A square matrix A is said to be unitary iff:

$$A^\theta = A^{-1}$$

   Multiplying both sides by A, we get an alternate definition of unitary matrix as given below:
   A square matrix A is said to be unitary iff:

$$AA^\theta = I = A^\theta A$$

   Example: $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is an example of a unitary matrix.

ILLUSTRATIVE EXAMPLES FROM GATE

Q.4 Real matrices $[A]_{3\times1}$, $[B]_{3\times3}$, $[C]_{3\times5}$, $[D]_{5\times3}$, $[E]_{5\times5}$ and $[F]_{5\times1}$ are given. Matrices [B] and [E] are symmetric.
Following statements are made with respect to these matrices.
1. Matrix product $[F]^T [C]^T [B] [C] [F]$ is a scalar.
2. Matrix product $[D]^T [F] [D]$ is always symmetric.

With reference to above statements, which of the following applies?

(a) Statement 1 is true but 2 is false
(b) Statement 1 is false but 2 is true
(c) Both the statements are true
(d) Both the statements are false

[CE, GATE-2004, 1 mark]

Solution: (a)

Statement 1 is true as shown below.
$[F]^T$ has a size 1×5
$[C]^T$ has a size 5×3
[B] has a size 3×3
[C] has a size 3×5
[F] has a size 5×1
So $[F]^T [C]^T [B] [C] [F]$ has a size 1×1. Therefore it is a scalar.
So, Statement 1 is true.

Consider Statement 2: $D^T F D$ is always symmetric.
Now $D' F D$ does not exist since $D'_{3 \times 5}$, $F_{5 \times 1}$ and $D_{5 \times 3}$ are not compatible for multiplication since, $D'_{3 \times 5} F_{5 \times 1} = X_{3 \times 1}$ and $X_{3 \times 1} D_{5 \times 3}$ does not exist.
So, Statement 2 is false.

Q.5 [A] is square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and difference of these matrices are defined as $[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$, respectively. Which of the following statements is **TRUE**?
(a) Both [S] and [D] are symmetric
(b) Both [S] and [D] are skew-symmetric
(c) [S] is skew-symmetric and [D] is symmetric
(d) [S] is symmetric and [D] is skew-symmetric

[CE, GATE-2007, 1 mark]

Solution: (d)

Since $S^t = (A + A^t)^t = A^t + (A^t)^t$
$= A^t + A = S$
i.e. $S^t = S$
$\therefore$ S is symmetric
Since $D^t = (A - A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t) = -D$
i.e. $D^t = -D$
So D is Skew-Symmetric.

Q.6 A square matrix B is skew-symmetric if
(a) $B^T = -B$
(b) $B^T = B$
(c) $B^{-1} = B$
(d) $B^{-1} = B^T$

[CE, GATE-2009, 1 mark]

Solution: (a)

A square matrix B is defined as skew-symmetric if and only if $B^T = -B$, by definition.

Q.7 Given the matrices $J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and

$K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, the product $K^T JK$ is _____.

[CE, GATE-2014 : 1 Mark, Set-1]

Solution:

$$J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$K^T JK = \begin{bmatrix}1 & 2 & -1\end{bmatrix}\begin{bmatrix}3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 6\end{bmatrix}\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$$

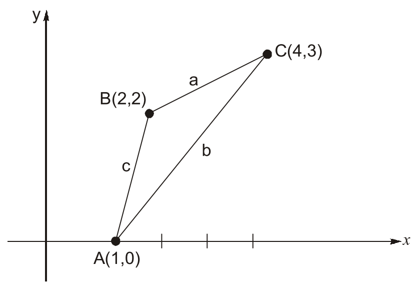
$$= \begin{bmatrix}6 & 8 & -1\end{bmatrix}\begin{bmatrix}1\\ 2\\ -1\end{bmatrix} = 6+16+1=23$$

Q.8 With reference to the conventional Cartesian (x, y) coordinate system, the vertices of a triangle have the following coordinates; $(x_1, y_1) = (1, 0)$; $(x_2, y_2) = (2, 2)$; $(x_3, y_3) = (4, 3)$. The area of the triangle is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$

(c) $\frac{4}{5}$ (d) $\frac{5}{2}$ **[CE, GATE-2014 : 1 Mark, Set-1]**

Solution : (a)



$$\text{Area of the triangle} = \frac{1}{2}\left|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right|$$

$$= \frac{1}{2}\left|1(2-3)+2(3-0)+4(0-2)\right| = \frac{1}{2}\left|-1+6-8\right| = \frac{3}{2}$$

Q.9 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I	List-II
A. Singular matrix	**1.** Determinant is not defined
B. Non-square matrix	**2.** Determinant is always one
C. Real symmetric	**3.** Determinant is zero
D. Orthogonal matrix	**4.** Eigenvalues are always real
	5. Eigenvalues are not defined

Codes:

	A	B	C	D
(a)	3	1	4	2
(b)	2	3	4	1
(c)	3	2	5	4
(d)	3	4	2	1

[ME, GATE-2006, 2 marks]

Solution: (a)

A. Singular matrix $\rightarrow$ Determinant is zero
B. Non-square matrix $\rightarrow$ Determinant is not defined
C. Real symmetric $\rightarrow$ Eigen values are always real
D. Orthogonal matrix $\rightarrow$ Determinant is always one

Q.10 Which one of the following equations is a correct identity for arbitrary 3×3 real matrices P, Q and R?

(a) $P(Q + R) = PQ + RP$
(b) $(P - Q)^2 = P^2 - 2PQ + Q^2$
(c) $\det(P + Q) = \det P + \det Q$
(d) $(P + Q)^2 = P^2 + PQ + QP + Q^2$

[ME, GATE-2014 : 1 Mark, Set-1]

Solution : (d)

$$(P + Q)^2 = P^2 + PQ+QP + Q^2 = P.P + P.Q + Q.P + Q.Q = P^2 + PQ + QP + Q^2$$

Q.11 Which one of the following statements is true for all real symmetric matrices?

(a) All the eigenvalues are real
(b) All the eigenvalues are positive.
(c) All the eigenvalues are distinct
(d) Sum of all the eigenvalues is zero.

[EE, GATE-2014 : 1 Mark, Set-2]

Answer : (a)

Q.12 Given an orthogonal matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $\left[AA^T\right]^{-1}$ is

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

[EC, GATE-2005, 2 marks]

Solution: (c)

For orthogonal matrix $AA^T = I$ i.e. Identity matrix.

$\therefore \quad (AA^T)^{-1} = I^{-1} = I$

Q.13 For matrices of same dimension M, N and scalar c, which one of these properties DOES NOT ALWAYS hold?

(a) $(M^T)^T = M$
(b) $(cM)^T = c(M)^T$
(c) $(M + N)^T = M^T + N^T$
(d) $MN = NM$

[EC, GATE-2014 : 1 Mark, Set-1]

Solution : (d)

Matrix multiplication is not commutative.

Q.14 Which one of the following statements is NOT true for a square matrix A?
(a) If A is upper triangular, the eigenvalues of A are the diagonal elements of it
(b) If A is real symmetric, the eigenvalues of A are always real and positive
(c) If A is real, the eigenvalues of A and A^T are always the same
(d) If all the principal minors of A are positive, all the eigenvalues of A are also positive

[EC, GATE-2014 : 2 Marks, Set-3]

Answer : (b)

Q.15 A real square matrix A is called skew-symmetric if
(a) $A^T = A$ (b) $A^T = A^{-1}$
(c) $A^T = -A$ (d) $A^T = A + A^{-1}$

[ME, 2016 : 1 Mark, Set-3]

Solution: (c)
A is skew-symmetric
$\therefore \quad A^T = -A$

Q.16 Let $M^4 = I$, (where I denotes the identity matrix) and $M \neq I$, $M^2 \neq I$ and $M^3 \neq I$. Then, for any natural number k, M^{-1} equals:
(a) M^{4k+1} (b) M^{4k+2}
(c) M^{4k+3} (d) M^{4k}

[EC, 2016 : 1 Mark, Set-1]

Solution: (c)
Given that $M^4 = I$ or $M^{4k} = I$ or $M^{4(k+1)} = I$
$\therefore \quad M^{-1} \times I = M^{4(k+1)} \times M^{-1}$
$\therefore \quad M^{-1} = M^{4k+3}$

1.3 DETERMINANTS

1.3.1 Definition

Let a_{11}, a_{12}, a_{21}, a_{22} be any four numbers. The symbol $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ represents the number $a_{11}a_{22} - a_{12}a_{21}$ and is called determinants of order 2. The number a_{11}, a_{12}, a_{21}, a_{22} are called elements of the determinant and the number $a_{11}a_{22} - a_{21}a_{12}$ is called the value of determinant.

1.3.2 Minors and Cofactors

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Leaving the row and column passing through the elements a_{ij}, then the second order determinant thus obtained is called the minor of element a_{ij} and we will be denoted by M_{ij}.

Example: The Minor of element $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

Similarly Minor of element $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$