15 *Years*

Previous Years Solved Papers

Civil Services Main Examination

(2009-2023)

Mathematics

Paper-I

Topicwise Presentation

MADE EASY
PUBLICATIONS

MADE EASY Publications Pvt. Ltd.

Corporate Office: 44-A/4, Kalu Sarai (Near Hauz Khas Metro Station), New Delhi-110016
E-mail: infomep@madeeasy.in
Contact: 9021300500
Visit us at: www.madeeasypublications.org

Civil Services Main Examination Previous Solved Papers : Mathematics Paper-I

First Edition: 2020

Second Edition: 2021

Third Edition: 2022

Fourth Edition: 2023

Fifth Edition: 2024

Preface

B. Singh (Ex. IES)

Civil Service is considered as the most prestigious job in India and it has become a preferred destination by all engineers. In order to reach this estimable position every aspirant has to take arduous journey of Civil Services Examination (CSE). Focused approach and strong determination are the pre-requisites for this journey. Besides this, a good book also comes in the list of essential commodity of this odyssey.

I feel extremely glad to launch the fourth edition of such a book which will not only make CSE plain sailing, but also with 100% clarity in concepts.

MADE EASY team has prepared this book with utmost care and thorough study of all previous years papers of CSE. The book aims to provide complete solution to all previous years questions with accuracy.

I would like to acknowledge efforts of entire MADE EASY team who worked day and night to solve previous years papers in a limited time frame and I hope this book will prove to be an essential tool to succeed in competitive exams and my desire to serve student fraternity by providing best study material and quality guidance will get accomplished.

With Best Wishes
B. Singh (Ex. IES)
CMD, MADE EASY Group

Previous Years Solved Papers of

Civil Services Main Examination

Mathematics : Paper-I

CONTENTS

Sl.	TOPIC	PAGE No.

1 Linear Algebra

1. Vector Space Over R and C

1.1 Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in R^4 which satisfy the equations $x_1 + x_2 + 2x_3 + x_4 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$ is a subspace of R^4. What is the dimension of this subspace. Find one of its bases.

(2009 : 12 Marks)

Solution:

ILD for vertical reaction at A;

$$V = \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 + x_2 + 2x_3 + x_4 = 0, 2x_1 + 3x_2 - x_3 + x_4 = 0\}$$

Then clearly $(0, 0, 0, 0) \in V$ and so V is non-empty.

Again let $x = (x_1, x_2, x_3, x_4) \in V$ and $y = (y_1, y_2, y_3, y_4) \in V$, Also let $\alpha, \beta \in R$.

$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4) \in R^4$$

Also, $\alpha(x_1 + x_2 + 2x_3 + x_4) + \beta(y_1 + y_2 + 2y_3 + y_4) = 0$

$\Rightarrow \alpha x + \beta y$ satisfies $x_1 + x_2 + 2x_3 + x_4 = 0$

Similarly $\alpha x + \beta y$ satisfies 2nd equation as well.

$$\therefore \qquad \alpha x + \beta y \in V$$

Dimension of the Subspace :

Any element of V is a solution to equation

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, its dimension is same as rowspace of coefficient matrix, i.e., its rank.

Reducing it to row reduced echelon form

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -5 & -1 \end{bmatrix}$$

As it has two non-zero rows in row reduced form.

$$\dim(v) = \text{Rank of matrix} = 2$$

Writing the equation as matrix

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } Ax = 0.$$

So, $\qquad V = \{x = (x_1, x_2, x_3, x_4) \in \text{R4} \mid Ax = 0\}$

Clearly $\qquad O = (0, 0, 0, 0) \in V$ so V is non-empty.

Let $\qquad x = (x_1, x_2, x_3, x_4)$; $y = (y_1, y_2, y_3, y_4) \in V$ and $\alpha, \beta \in R$

$$A(\alpha x + by) = A(\alpha x) + A(\beta y)$$

$$= \alpha(Ax) + \beta(Ay) = \alpha\cdot 0 + \beta\cdot 0 = 0$$

$\therefore \quad \alpha x + \beta y \in V$

$\therefore$ V is a vector subspace.

Clearly, V is the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$

Reducing it to row reduced echelon form

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{R_L \leftrightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -5 & -1 \end{bmatrix}$$

$$\dim(\text{Row space } (A)) = \text{Number of non-zero rows} = 2$$

By Rank-Nullity Theorem :

$$\underset{(\text{Nullity})}{\dim(\text{Null space})} + \underset{(\text{Rank})}{\dim(\text{Row space})} = n = 4$$

$$\dim(\text{null space}) = 2$$

$\therefore \quad \dim(V) = 2$

For Finding Basis :

$$\dim(\text{null space}) = 2$$

$\therefore$ No. of free variables $= 4 - 2 = 2$

So, we fix 2 variables.

Taking $x_3 = 1$, $x_4 = 0$ first.

$$\left.\begin{aligned} x_1 + x_2 &= -2 \\ 2x_1 + 3x_2 &= 1 \end{aligned}\right\} \begin{aligned} x_1 &= -7 \\ x_2 &= 5 \end{aligned} \qquad x = (-7, 5, 1, 0)$$

Taking $x_3 = 0$, $x_4 = 1$

$$\left.\begin{aligned} x_1 + x_2 &= -1 \\ 2x_1 + 3x_2 &= -1 \end{aligned}\right\} \begin{aligned} x_1 &= -2 \\ x_2 &= 1 \end{aligned} \qquad x = (-2, 1, 0, 1)$$

$\therefore$ $(-7, 5, 1, 0)$ and $(-2, 1, 0, 1)$ are two elements of V. And since they are linearly independent **(because of choice of 3rd and 4th element)** they form a basis.

1.2 **Prove that set V of all 3×3 real symmetric matrices form a linear subspace of the space of all 3×3 real matrices. What is the dimension of this subspace? Find at least one of the bases for V.**

(2009 : 20 Marks)

Solution:

Approach : Use definition of subspaces for first part. For the 2nd impose conditions due to symmetricity on the matrix.

Let V be subset of all 3×3 symmetric matrix.

Then $I_3 \in V$ so V is not empty. Again, let $A, B \in V$.

$\Rightarrow \quad A = A^T$

$B = B^T$ (definition of symmetric)

and $\alpha, \beta \in R$.

Then
$$(\alpha A + \beta B)^T = (\alpha A)^T + (\beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B$$

$\therefore$ $\alpha A + \beta B \in V$.

So, V is a vector subspace of the space of all 3×3 real matrices over R.

2 Calculus

1. Function of a Real Variable

1.1 Suppose that f'' is continuous on [1, 2] and that f has three zeros in the interval (1, 2). Show that f'' has at least one zero in the interval (1, 2).

(2009 : 12 Marks)

Solution:

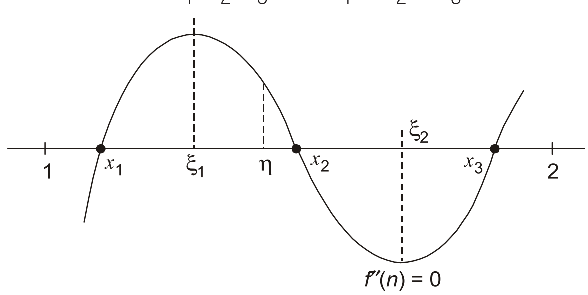
Insight : This question uses the fact that **continuity of any derivative of a function ensures continuity and differentiability of lower order derivatives** and the Rolle's theorem.

f'' is continuous on [1, 2]

$\Rightarrow f'$ is continuous and differentiable on [1, 2]

$\Rightarrow f$ is continuous and differentiable on [1, 2]

f has three zeros in (1, 2). Let them be x_1, x_2, x_3 with $x_1 < x_2 < x_3$.



Inflection Point

In the interval $[x_1, x_2]$, applying Rolle's theorem.

f is continuous on $[x_1, x_2]$.

f is differentiable on (x_1, x_2).

$$f(x_1) = f(x_2) = 0$$

$\Rightarrow \exists\, \xi_1 \in (x_1, x_2)$ such that $f'(\xi) = 0$ by Rolle's theorem.

Similarly, applying Rolle's theorem in interval $[x_2, x_3]$, $\exists\, \xi_2 \in (x_2, x_3)$ such that $f'(\xi_2) = 0$.

As $\xi_1 < x_2$ and $\xi_2 > x_2 \Rightarrow \xi_1 < \xi_2$.

Applying Rolle's theorem on f' in (ξ_1, ξ_2).

f' is continuous on $[\xi_1, \xi_2]$.

f' is differentiable on (ξ_1, ξ_2) as f'' is continuous on that interval.

$$f'(\xi_1) = f'(\xi_2) = 0$$

$\Rightarrow \exists\, \eta \in (\xi_1, \xi_2)$ so that $f''(\eta) = 0$ by Rolle's theorem.

Also, $(x_1, x_2) \subset (1, 2) \Rightarrow \eta \in (1, 2)$

1.2 If f is the derivative of some function defined on $[a, b]$ prove that there exists a number $\eta \in [a, b]$ such that

$$\int_a^b f(t)\,dt = f(\eta)(b-a)$$

(2009 : 12 Marks)

Solution:

Insight : This is the mean value theorem of integral calculus with the difference that ***f* is not given as continuous but as derivative of some function**. We use 2nd Fundamental Theorem of Calculus.

f is derivative on some function

$$\Rightarrow \quad f(x) = F(x) \text{ on } [a, b]$$

i.e., $f(x)$ has an anti-derivative $F(x)$ defined on $[a, b]$.

By 2nd Fundamental Theorem of Algebra, for any $x_1, x_2 \in [a, b]$

$$\int_{x_1}^{x_2} f(x)dx = F(x_2) - F(x_1)$$

Proof of 2nd Fundamental Theorem : $F(x)$ is an anti-derivative of $f(x)$.

Let $$G(x) = \int_{x_1}^{x} f(x)dx \text{ where } x_1 \in (a, b)$$

Then $G(x)$ is an anti-derivative.

As **two anti-derivatives differ by a constant**

$$G(x) = F(x) + C$$

$$\therefore \quad G(x_1) = F(x_1) + C$$

and $$\int_{x_1}^{x_1} f(x) = G(x_1) = 0$$

$$\Rightarrow \quad F(x_1) + C = 0 \Rightarrow C = -F(x_1)$$

$$\therefore \quad G(x) = F(x) - F(x_1)$$

$$\therefore \quad \int_{x_1}^{x} f(x)dx = F(x) - F(x_1)$$

$$\Rightarrow \quad \int_{x_1}^{x_2} f(x)dx = F(x_2) - F(x_1)$$

Now, $F'(x) = f(x)$ on $[a, b]$ so $F(x)$ is continuous on $[a, b]$ as **every differentiable function is continuous** and $F(x)$ is differentiable on (a, b).

So by Mean Value Theorem, $\exists \eta \in (a,b)$ such that

$$F'(\eta) = \frac{F(b) - F(a)}{b - a}$$

$$\Rightarrow \quad F(b) - F(a) = (b - a)f(\eta)$$

$$\Rightarrow \quad \int_a^b f(x)dx = (b - a)f(\eta)$$

1.3 **A twice differentiable function $f(x)$ is such that $f(a) = 0 = f(b)$ and $f(c) > 0$ for $a < c < b$. Prove that there is at least one point x, $a < \xi < b$, for which $f''(\xi) < 0$.**

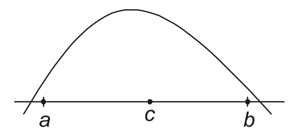
(2010 : 12 Marks)

Solution:

Given $f(a) = f(b) = 0$ and $c \in (a, b)$ such that $f(c) > 0$.

By Lagrange's Mean Value Theorem (LMVT), $\exists\ \alpha \in (a,c)$ and $\beta \in (c,b)$:

$$f'(\alpha) = \frac{f(c) - f(a)}{c - a} \text{ and } f'(\beta) = \frac{f(b) - f(c)}{b - c} \quad ...(1)$$



Now let $\xi \in (\alpha, \beta)$

$$\therefore \quad \xi \in (a, b) \text{ or } a < \xi < b$$

By Lagrange's Mean Value Theorem (LMVT)

$$f''(\xi) = \frac{f'(\beta) - f'(\alpha)}{\beta - \alpha}$$

Using values in eqn. (1), we get

$$f''(\xi) = \frac{\frac{f(b)-f(c)}{b-c} - \frac{f(c)-f(a)}{c-a}}{\beta - \alpha}$$

$$\Rightarrow \quad f''(\xi) = \frac{-\frac{f(c)}{b-c} - \frac{f(c)}{c-a}}{\beta - \alpha} \qquad [f(a) = f(b) = 0]$$

$$\Rightarrow \quad f''(\xi) = -\left[\frac{\frac{f(c)}{b-c} + \frac{f(c)}{c-a}}{\beta - \alpha}\right] < 0 \text{ as } \beta > \alpha,\ b > c,\ c > a$$

$\therefore \exists\, \xi$ such that $\quad f''(\xi) < 0$, where $a < \xi < b$.

1.4 **Show that a box (rectangular parallelopiped) of maximum volume V with prescribed surface area is a cube.**

(2010 : 20 Marks)

Solution:

Let x, y, z be length of edges of given rectangular parallelopiped. Its surface area be S, volume be V. (x, y, $z \neq 0$).

$$\therefore \quad S = 2xy + 2yz + 2zx,\ V = xyz$$

Let λ be the Lagrange's multiplier ($\lambda \neq 0$)

So, $\quad f = xyz + \lambda(2xy + 2yz + 2xz - S)$

$\therefore$ At extremum,

$$f_x = 0 \Rightarrow yz + \lambda(2y + 2z) = 0 \qquad ...(1)$$

$$f_y = 0 \Rightarrow xz + \lambda(2x + 2z) = 0 \qquad ...(2)$$

$$f_z = 0 \Rightarrow xy + \lambda(2y + 2x) = 0 \qquad ...(3)$$

Multiplying (1) by x and (2) by y and subtracting then, we get

$$xyz + \lambda(2xy + 2xz) - xyz - \lambda(2xy + 2yz) = 0$$

$$\Rightarrow \quad \lambda(2xz - 2yz) = 0$$

$$\Rightarrow \quad 2xz - 2yz = 0 \qquad (\lambda \neq 0)$$

$$\Rightarrow \quad x = y \text{ as } z \neq 0 \qquad ...(4)$$

Similarly, multiplying (2) by y and (3) by z and subtracting, we get

$$xyz + \lambda(2xy + 2yz) - xyz - \lambda(2yz + 2xz) = 0$$

$$\Rightarrow \quad \lambda(2xy - 2xz) = 0$$

$$\Rightarrow \quad y = z\ (x \neq 0) \qquad ...(5)$$

$\therefore$ from (4) and (5), it can be concluded $x = y = z$ which is a cube.

1.5 **Let f be a function defined on $\mathbb{R}$ such that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x in $\mathbb{R}$. How long can $f(2)$ possibly be?**

(2011 : 10 Marks)

3 Analytic Geometry

1. Straight Lines

1.1 **A line is drawn through a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$ to meet fixed lines $y=mx$, $z=c$ and $y=-mx$, $z=-c$. Find the locus of the line.**

(2009 : 12 Marks)

Solution:

Approach : Use general equation of line intersecting two lines given in planar form.

Given fixed lines are

$$y - mx = 0,\ z - c = 0 \qquad \text{...(i)}$$

$$y + mx = 0,\ z + c = 0 \qquad \text{...(ii)}$$

General equation of line intersecting both

$$(y - mx) + k_1(z - c) = 0 = (y + mx) + k_2(z + c) \qquad \text{...(iii)}$$

If it meets ellipse we eliminate k_1 and k_2

Putting $z = 0$ in (iii)

$$y - mx - k_1c = 0;\ y + mx + k_2c = 0$$

$$\Rightarrow \qquad \frac{y}{-k_2m+k_1m} = \frac{x}{-(k_1+k_2)} = \frac{c}{2m}$$

$$\Rightarrow \qquad x = \frac{-(k_1+k_2)c}{2m};\ y = \frac{(k_1-k_2)c}{2}$$

Putting this in equation of ellipse

$$\frac{(k_1+k_2)^2c^2}{4m^2a^2}+\frac{(k_1-k_2)^2c^2}{4b^2} = 1$$

$$(k_1 + k_2)^2c^2b^2 + (k_1 - k_2)^2c^2a^2m^2 = 4a^2b^2m^2$$

Substituting k_1 and k_2 from (iii)

$$\left\{\left(\frac{mx-y}{z-c}\right)+\left(-\frac{mx+y}{z+c}\right)\right\}^2 c^2b^2 + \left\{\left(\frac{mx-y}{z-c}\right)+\left(\frac{mx+y}{z+c}\right)\right\}^2 \times c^2a^2 = 4a^2b^2m^2$$

$$\Rightarrow [(mx - y)(z + c) - (mx + y)(z - c)]^2c^2b^2 + [(mx - y)(z + c) + (mx + y)(z - c)]^2m^2c^2a^2 = 4a^2b^2m^2(z^2 - c^2)^2$$

$$\Rightarrow [cmx - yz]^2\ c^2b^2 + [mxz - cy]^2m^2c^2a^2 = a^2b^2m^2(z^2 - c^2)^2$$

which is required locus.

1.2 **Prove that two of the straight lines represented by the equation**

$$x^3 + bx^2y + cxy^2 + y^3 = 0$$

will be at right angles, if $b + c = -2$.

(2012 : 12 Marks)

Solution:

The given equation is a homogeneous equation of third degree and hence it represents three straight lines through the origin.

Let $y = mx$ be any of these lines.

Replacing $\frac{y}{x}$ by m in $x^3 + bx^2y + cxy^2 + y^3 = 0$ or $1+b\frac{y}{x}+c\frac{y^2}{x^2}+\frac{y^3}{x^3} = 0$, we get

$$m^3 + cm^2 + bm + 1 = 0 \quad \text{...(i)}$$

Let m_1, m_2, m_3 be its roots, then

$$m_1 \cdot m_2 \cdot m_3 = -1$$

But, two of these lines, say with slopes, m_1 and m_2, are at right angles,

then, $\quad m_1 \cdot m_2 = -1$

Thus, $\quad (-m_3) = 1$ or $m_3 = 1$

But m_3 is a root of (i)

$\therefore \quad 1 + c + b + 1 = 0$

or $\quad b + c = -2$

1.3 Verify if the lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar. If yes, then find the equation of the plane in which they lie?

(2014 : 7 Marks)

Solution:

Two straight lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

And equation of plane containing them, is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Here, in our case,

$$\begin{vmatrix} (b-c)-(a-d) & b-a & b+c-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\delta \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_2 \\ C_3 \to C_3 - C_2 \end{matrix}$$

$$= \begin{vmatrix} d-c & b-a & c-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix} = 0 \text{ as } C_1 = -C_3$$

Hence, the given lines are coplanar.

The equation of the plane containing them, is

$$\begin{vmatrix} x-(a-d) & y-a & z-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0. \text{ Applying } \begin{matrix} C_1 \to C_1 - C_2 \\ C_3 \to C_3 - C_2 \end{matrix}$$

$$\begin{vmatrix} x-y+d & y-a & z-y-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2y+z & y-a & z-y-d \\ 0 & \alpha & \delta \\ 0 & \beta & \gamma \end{vmatrix} = 0 \text{ as } C_1 \to C_1 + C_3$$

$\Rightarrow \quad x - 2y + z = 0$

4 Ordinary Differential Equations

1. Formulation of Differential Equations

1.1 Find the differential equations of the family of circles in the xy-plane passing through (–1, 1) and (1, 1).

(2009 : 20 Marks)

Solution:

Approach : First use conditions to get the general equation of such a circle. Then get the differential equations.

General equation of circle in xy plane is

$$x^2 + y^2 + 2ax + 2by + d = 0 \qquad ...(i)$$

It passes through (–1, 1) and (1, 1)

$$\Rightarrow \quad 2 - 2a + 2b + d = 0 \Rightarrow 4a = 0$$

$$2 + 2a + 2b + d = 0 \Rightarrow d = -(2b + 2)$$

$\therefore$ General equation of circles passing through (–1, 1) and (1, 1) is

$$x^2 + y^2 + 2by - (2b + 2) = 0 \qquad ...(ii)$$

where b is the single parameter.

Differentiating (ii) with respect to x

$$2x + 2y\frac{dy}{dx} + 2b\frac{dy}{dx} = 0$$

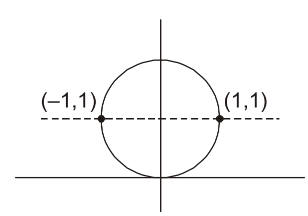
$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{b+y} \Rightarrow b = \frac{-x}{dy/dx} - y$$

Putting this in (ii)

$$x^2 + y^2 + 2\left(\frac{-x}{dy/dx} - y\right)y - \left[2\left(\frac{-x}{dy/dx} - y\right) + 2\right] = 0$$

$$\Rightarrow \quad x^2 - y^2 - \frac{2xy}{dy/dx} + \frac{2x}{dy/dx} + 2y - 2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2 - 2y + 2}{2x(1-y)}$$



which is the required differential equation.

Alternatively : We can also use equation of circle $x^2 + (y - 1)^2 + \lambda(y - 1) = 0$ and proceed.

1.2 **Find the curve for which the part of the tangent cut-off by the axes is bisected at the point of tangency.**

(2014 : 10 Marks)

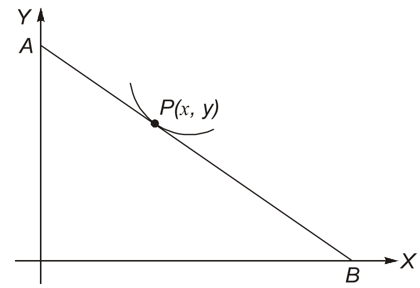
Solution:

Let equation of tangent line at point 'P' at

$$\frac{X - x}{Y - y} = \frac{dy}{dx} \qquad ...(i)$$

Now, its point of intersection with co-ordinate axes are

$$A\left(0, y - \frac{x}{\frac{dy}{dx}}\right);\ B\left(x - y\frac{dy}{dx}, 0\right).$$



Given : 'P' is mid point of AB.

So, $$\frac{x - y\frac{dy}{dx}}{2} = x \text{ and } \frac{\left(y - \frac{x}{\frac{dy}{dx}}\right)}{2} = y$$

$\Rightarrow$ $$x = -y\frac{dy}{dx} \text{ and } y = -\frac{x}{\frac{dy}{dx}}$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{x}{y}$$

$\Rightarrow$ $$ydy + xdx = 0$$

Integrating, we get

$$x^2 + y^2 = C \text{ which the required curve.}$$

1.3 Find the differential equation (DE) representing all the circles in the xy-plane.

(2017 : 10 Marks)

Solution:

Method I : General equation of circle

$$(x-a)^2 + (y-b)^2 = r^2$$

Differentiating w.r.t. x,

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

i.e., $$(x-a) + (y-b)y_1 = 0 \quad \text{...(i)}$$

Differentiating again w.r.t. x

$$1 + (y-b)y_2 + y_1^2 = 0 \quad \text{...(ii)}$$

Differentiating again w.r.t. x

$$(y-b)y_3 + y_1y_2 + 2y_1y_2 = 0$$

i.e., $$(y-b) = \frac{-3y_1y_2}{y_3}$$

Substituting it in (ii)

$$1 + \left(\frac{-3y_1y_2}{y_3}\right)y_2 + y_1^2 = 0$$

i.e., $$(1+y_1^2)y_3 - 3y_1y_2^2 = 0$$

i.e., $$(1+y_1^2)y_3 = 3y_1y_2^2$$

Method II : Using curvature-formula (K).

1.4 Find the orthogonal trajectories of the family of circles passing the points (0, 2) and (0, –2).

(2020 : 10 Marks)

Solution:

Let equation of circle through (0, 2) and (0, –2) be:

$$x^2 + (y^2 - 4) + \lambda x = 0, \quad \lambda : \text{Parameter} \qquad ...(1)$$

Differentiating w.r.t. x, we get:

$$2x + 2y\left(\frac{dy}{dx}\right) + \lambda = 0 \qquad ...(2)$$

From (1) and (2),

$$x^2 + y^2 - 4 + \left[-2x - 2y\left(\frac{dy}{dx}\right)\right]x = 0$$

$$y^2 - x^2 - 4 - 2xy\frac{dy}{dx} = 0 \qquad ...(3)$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3), we get

$$y^2 - x^2 - 4 + 2xy\frac{dy}{dx} = 0$$

$$\frac{(y^2-4)dy}{y^2} + \frac{2xy\,dx - x^2dy}{y^2} = 0$$

$$\int\left(1 - \frac{4}{y^2}\right)dy + \int d\left(\frac{x^2}{y}\right) = 0$$

$$y + \frac{4}{y} + \frac{x^2}{y} = c$$

$\Rightarrow$ $$x^2 + y^2 + 4 = cy \text{ (required trajectory)}$$

1.5 Find the orthogonal trajectories of the family of confocal conics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1; a > b > 0$$

are constants and λ is a parameter. Show that the given family of curves is self orthogonal.

[2021 : 10 marks]

Solution:

(i)

Given: $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ ... (i)

Differentiating (i), $\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$

or $x\left(b^2+\lambda\right) + y\left(a^2+\lambda\right)\frac{dy}{dx} = 0$

or, $$\lambda\left(x + y\frac{dy}{dx}\right) = -\left(b^2x + a^2y\frac{dy}{dx}\right)$$

$\therefore$ $$\lambda = -\left[\frac{b^2x + a^2y\left(\frac{dy}{dx}\right)}{x + y\left(\frac{dy}{dx}\right)}\right]$$

$\therefore$ $$a^2 + \lambda = a^2 - \frac{b^2x + a^2y\left(\frac{dy}{dx}\right)}{x + y\left(\frac{dy}{dx}\right)} = \frac{\left(a^2 - b^2\right)x}{x + y\left(\frac{dy}{dx}\right)}$$

5 Dynamics & Statics

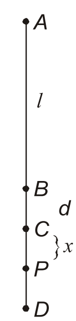
1. Rectilinear Motion

1.1 One end of a light elastic string of natural length l and modulus of elasticity 2 mg is attached to a fixed point O and the other end to a particle of mass m. The particle initially held at rest at O is bit fall. Find the greatest extension of the string during motion and show that the particle will reach O again after a time.

$$(\pi + 2 - \tan^{-1} 2)\sqrt{\frac{2l}{g}}$$

(2009 : 20 Marks)

Solution:



Let C be the equilibrium position of the body and $BC = d$.

In position of equilibrium

$$mg = 2mg \cdot \frac{d}{l} \quad \Rightarrow \quad d = \frac{l}{2}$$

When particle is dropped from A it free falls till B.

$$V_B^2 = 0 + 2g \times AB$$

$$\Rightarrow \quad V_B = \sqrt{2gl}$$

After B the tension in the string starts acting which is balanced at C. Beyond C the particles moves due to its velocity till it comes to stop at D.

At any point P with $CP = x$.

$$m\frac{d^2x}{dt^2} = mg - (2mg)\frac{d+x}{l}$$

$$= -2mg\frac{x}{l}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{2g}{l}x$$

So, the body performs SHM with centre C.

Multiplying with $2\frac{dx}{dt}$ and integrating

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{l}x^2 + C$$

At B, $\quad V_B = \sqrt{2gl}, \; x = -\frac{l}{2}$

$$\therefore \quad 2gl = -\frac{2g}{l}\frac{l^2}{4} + C \quad \Rightarrow \quad C = \frac{5}{2}gl$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{5}{2}gl - \frac{2g}{l}x^2 \qquad \text{...(i)}$$

At D, $$\frac{dx}{dt} = 0$$

$$\Rightarrow \quad x^2 = \frac{5}{4}l^2 \quad \Rightarrow \quad x = \frac{\sqrt{5}}{2}l$$

So greatest distance through which particle falls

$$= AD = AB + BC + CD$$

$$= l + \frac{l}{2} + \frac{\sqrt{5}l}{2} = \frac{(3+\sqrt{5})l}{2}$$

Greatest extension $= \dfrac{(1+\sqrt{5})l}{2}$

From (i), $$\frac{dx}{dt} = \sqrt{\frac{2g}{l}}\left[\frac{5}{4}l^2 - x^2\right]^{1/2}$$

where positive sign is taken as particle is moving in direction of increasing x.

$$\sqrt{\frac{l}{2g}}\frac{dx}{\sqrt{\frac{5}{4}l^2 - x^2}} = dt$$

If t_1 is time from B to D

$$\sqrt{\frac{l}{2g}}\int_{-d/2}^{\sqrt{5}l/2}\frac{dx}{\sqrt{\frac{5}{4}l^2 - x^2}} = \int_0^t dt$$

$$\Rightarrow \quad \sqrt{\frac{l}{2g}}\left[\sin^{-1}\frac{x}{\sqrt{5}l/2}\right]_{-l/2}^{\sqrt{5}l/2} = t_1$$

$$t_1 = \sqrt{\frac{l}{2g}}\left[\frac{\pi}{2} - \sin^{-1} - \frac{1}{\sqrt{5}}\right] = \sqrt{\frac{l}{2g}}\left[\frac{\pi}{2} + \sin\frac{1}{\sqrt{5}}\right]$$

$$= \sqrt{\frac{l}{2g}}\left[\frac{\pi}{2} + \tan^{-1}\frac{1}{2}\right]$$

$$= \sqrt{\frac{l}{2g}}\left[\frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1}2\right]$$

$$= \sqrt{\frac{l}{2g}}\left[\pi - \tan^{-1}2\right]$$

Time in falling from A to B.

$$\frac{1}{2}gt_2^2 = l \quad \Rightarrow \quad t_2 = \sqrt{\frac{2l}{g}}$$

$\therefore$ Total time taken to come back to

$$O = 2\sqrt{\frac{l}{2g}}[\pi - \tan^{-1}2 + 2]$$

$$= \sqrt{\frac{2l}{g}}[\pi + 2 - \tan^{-1}2]$$

6 Vector Analysis

1. Scalar and Vector Fields

1.1 Prove that the vectors $\vec{a} = 3\boldsymbol{i} + j - 2k$, $\vec{b} = -\boldsymbol{i} + 3j + 4k$, $\vec{c} = 4\boldsymbol{i} - 2j - 6k$ can form the sides of a triangle. Find the lengths of the medians of the triangle.

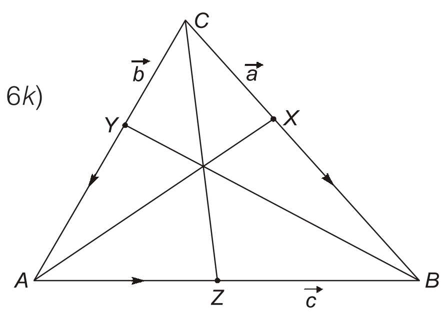
(2016 : 10 Marks)

Solution:

Here, we find that

$$\vec{b}+\vec{c} = (-i + 3j + 4k) + (4i - 2j - 6k)$$
$$= 3i + j - 2k$$
$$= \vec{a}$$

i.e., $\quad \vec{b}+\vec{c} = \vec{a}$



And also we notice that these three vectors are not collinear (components are not proportional). Hence, these form the sides of a triangle. Let AX, BY and CZ be medians.

By triangle law of vector addition :

$$\overrightarrow{AX} = \overrightarrow{AB}+\overrightarrow{BX} = \vec{C}-\frac{\vec{a}}{2} = \frac{1}{2}(5i-5j-14k)$$

$$\left|\overrightarrow{AX}\right| = \sqrt{\frac{1}{4}(25+25+196)} = \sqrt{\frac{246}{4}} = \frac{\sqrt{246}}{2}$$

$$\overrightarrow{BY} = \overrightarrow{BA}+\overrightarrow{AY} = -\left(\vec{C}+\frac{\vec{b}}{2}\right) = -\frac{1}{2}(7\hat{i}-\hat{j}-8\hat{k})$$

$$\left|\overrightarrow{BY}\right| = \frac{1}{2}\sqrt{49+1+64} = \frac{\sqrt{114}}{2}$$

$$\overrightarrow{CZ} = \overrightarrow{CA}+\overrightarrow{AZ} = \vec{b}+\frac{\vec{c}}{2} = \frac{1}{2}(2i+4j+2k)$$

$$\left|\overrightarrow{CZ}\right| = |i+2j+k| = \sqrt{1+4+1} = \sqrt{6}$$

1.2 If, $\vec{a} = \sin\theta\,\hat{\boldsymbol{i}} + \cos\theta\,\hat{j} + \theta\hat{k}$, $\vec{b} = \cos\theta\,\hat{\boldsymbol{i}} - \sin\theta\,\hat{j} - 3\hat{k}$, $\vec{c} = 2\hat{\boldsymbol{i}} + 3\hat{j} - 3\hat{k}$ then find the values of the derivative of the vector function $\vec{a}\times(\vec{b}\times\vec{c})$ w.r.t. θ at $\theta = \frac{\pi}{2}$ and $\theta = \pi$.

(2023 : 10 marks)

Solution:

Given that,

$$\vec{a} = \sin\theta\hat{i} + \cos\theta\hat{j} + \theta\hat{k}$$
$$\vec{b} = \cos\theta\hat{i} - \sin\theta\hat{j} - 3\hat{k}$$
$$\vec{c} = 2\hat{i} + 3\hat{j} - 3\hat{k}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} i & j & k \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix}$$

$$= i(3\sin\theta + 9) + j(-6 + 3\cos\theta) + k(3\cos\theta + 2\sin\theta)$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} i & j & k \\ \sin\theta & \cos\theta & \theta \\ 3\sin\theta+9 & -6+3\cos\theta & 3\cos\theta+2\sin\theta \end{vmatrix}$$

$$= i(3\cos^2\theta + 2\sin\theta\cos\theta + 6\,\theta - 3\,\theta\cos\theta) + j(3\theta\sin\theta + 9\theta - 3\sin\theta\cos\theta - 2\sin 2\theta)$$
$$+ K(-6\sin\theta + 3\sin\theta\cos\theta - 3\sin\theta\cos\theta - 9\cos\theta)$$

$$= i(3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta) + j(-2\sin^2\theta - 3\sin\theta\cos\theta + 3\theta\sin\theta + 9\theta)$$
$$+ K(-6\sin\theta - 9\cos\theta)$$

$$\frac{d}{d\theta}[\vec{a}\times(\vec{b}\times\vec{c})] = i[-6\sin\theta\cos\theta + 2\cos 2\theta - 3\cos\theta + 3\,\theta\sin\theta + 6];$$
$$+ j(-3(-\sin^2\theta + \cos^2\theta) - 4\sin\theta\cos\theta + 3(\theta\cos\theta + \sin\theta) + 9$$
$$+ K(-6\cos\theta + 9\sin\theta)$$

(i) At, $\theta = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$

$$\therefore \frac{d}{d\theta}[\vec{a}\times(\vec{b}\times\vec{c})] = \left(0+(-2)+\frac{3\pi}{2}+6\right)i + (3+0+3+9)j + (-6\times 0+9)k$$

$$= \left(\frac{3\pi}{2}+4\right)\hat{i} + 15\hat{j} + 9\hat{k}$$

(ii) At $\theta = \pi$

$$\frac{d}{d\theta}\left(\vec{a}\times(\vec{b}\times\vec{c})\right) = (0 + 2 + 3 + 6)i + (-3 - 3\pi + 9)j + (6 + 0)k$$

$$= 11i + (6-3\pi)\hat{j} + 6\hat{k}$$

$\therefore$ at $\theta = \pi$

$$\frac{d}{d\theta}\left(\vec{a}\times(\vec{b}\times\vec{c})\right) = 11i + (6-3\pi)j + 6\hat{k}$$

2. Differentiation of a Vector Field of a Scalar Variable

2.1 For two vectors $\vec{a}$ and $\vec{b}$ given respectively by

$$\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$$

and

$$\vec{b} = \sin t\hat{i} - \cos t\hat{j}$$

Determine : (i) $\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)$ and $\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)$

(2009 : 10 Marks)

Solution :

$$\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$$

$$\vec{b} = \sin 5t\,\hat{i} - \cos t\,\hat{j}$$

$$\vec{a}\cdot\vec{b} = 5t^2 \sin 5t - t\cos t \qquad (\because \hat{i}\cdot\hat{i} = 1 \text{ etc., } \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0)$$

$$\therefore \quad \frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d}{dt}(5t^2\sin 5t - t\cos t)$$

$$= 5(2t\sin 5t + t^2\cdot 5\cos 5t) - (1\cdot\cos t - t\sin t)$$

$$= 10t\sin 5t + 25t^2\cos 5t - \cos t + t\sin t$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin 5t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{i}(0 - t^3\cos t) + \hat{j}(-t^3\sin 5t - 0) + \hat{k}(-5t^2\cos t - t\sin 5t)$$

$$= -t^3\cos t\hat{i} - t^3\sin 5t\hat{j} - (5t^2\cos t + t\sin 5t)\hat{k}$$

$$\therefore \quad \frac{d}{dt}(\vec{a}\times\vec{b}) = (-3t^2\cos t + t^3\sin t)\hat{i} - (3t^2\sin 5t + 5t^3\cos t)\hat{j} -$$

$$(10t\cos t - 5t^2\sin t + t\cos 5t + 1\cdot\sin 5t)\hat{k}$$

2.2 If

$$\vec{A} = x^2yz\vec{i} - 2xz^3\vec{j} + xz^2\vec{k}$$

$$\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$$

find the value of $\dfrac{\partial^2}{\partial x\partial y}(\vec{A}\times\vec{B})$ at (1, 0, –2).

(2012 : 12 Marks)

Solution:

Given :

$$\vec{A} = x^2yz\vec{i} - 2xz^3\vec{j} + xz^2\vec{k}$$

$$\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$$

$$\therefore \quad \vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= \vec{i}(2x^3z^3 - xyz^2) + \vec{j}(2xz^3 + x^4yz) + \vec{k}(x^2y^2z + 4xz^4)$$

$$\frac{\partial}{\partial y}(\vec{A}\times\vec{B}) = \vec{i}(-xz^2) + \vec{j}(x^4z) + \vec{k}(2x^2yz)$$

$$\Rightarrow \quad \frac{\partial^2}{\partial x\partial y}(\vec{A}\times\vec{B}) = \vec{i}(-z^2) + \vec{j}(4x^3z) + \vec{k}(4xyz)$$

$\therefore$ At (1, 0, –2)

$$\frac{\partial^2}{\partial x\partial y}(\vec{A}\times\vec{B}) = -4\vec{i} - 8\vec{j}$$