15 *Years*

Previous Years Solved Papers

Civil Services Main Examination

(2009-2023)

Mathematics

Paper-II

Topicwise Presentation

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Civil Services Main Examination Previous Solved Papers : Mathematics Paper-II

First Edition: 2020

Second Edition: 2021

Third Edition: 2022

Fourth Edition: 2023

Fifth Edition: 2024

Preface

B. Singh (Ex. IES)

Civil Service is considered as the most prestigious job in India and it has become a preferred destination by all engineers. In order to reach this estimable position every aspirant has to take arduous journey of Civil Services Examination (CSE). Focused approach and strong determination are the pre-requisites for this journey. Besides this, a good book also comes in the list of essential commodity of this odyssey.

I feel extremely glad to launch the fourth edition of such a book which will not only make CSE plain sailing, but also with 100% clarity in concepts.

MADE EASY team has prepared this book with utmost care and thorough study of all previous years papers of CSE. The book aims to provide complete solution to all previous years questions with accuracy.

I would like to acknowledge efforts of entire MADE EASY team who worked day and night to solve previous years papers in a limited time frame and I hope this book will prove to be an essential tool to succeed in competitive exams and my desire to serve student fraternity by providing best study material and quality guidance will get accomplished.

With Best Wishes

B. Singh (Ex. IES)

CMD, MADE EASY Group

Previous Years Solved Papers of

Civil Services Main Examination

Mathematics : Paper-II

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1 Modern Algebra

1. Groups

1.1 If $\mathbb{R}$ is the set of real number and $\mathbb{R}_+$ is the set of positive real numbers, show that $\mathbb{R}$ under addition $(\mathbb{R}, +)$ and $\mathbb{R}_+$ under multiplication $(\mathbb{R}, +)$ are isomorphic. Similarly, if $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}_+$ the set of positive rational numbers, are $(\mathbb{Q}, +)$ and $(\mathbb{Q}_+, +)$ isomorphic? Justify your answer.

(2009 : 4+8=12 Marks)

Solution:

Let $\mathbb{R}$ be the set of real number and $\mathbb{R}_+$ be the set of positive real number.

We have to show

$$(\mathbb{R},+) \cong (\mathbb{R}_+,\cdot)$$

Define $f: \mathbb{R} \to \mathbb{R}_+$ as

$$f(x) = a^x;\text{ where } a > 0.$$

We will show f is one-one.

Consider,

$$\begin{aligned}\ker f &= \{x \in R \mid f(x) = 1\} \\ &= \left\{x \in \mathbb{R} \mid a^x = 1\right\} \\ &= \left\{x \in \mathbb{R} \mid x = \log_a^1\right\} \\ &= \left\{x \in \mathbb{R} \mid x = 0\right\} \\ &= \{0\}\end{aligned}$$

$\therefore$ f is 1 – 1.

We will show f is homomorphism.

Let $x, y \in R$

Consider

$$\begin{aligned}f(x + y) &= a^{x+y} \\ &= a^x.a^y = f(x).f(y)\end{aligned}$$

$\therefore$ f is homomorphism. We will show f is onto, i.e., we have to find for any positive real number 'y' some real number x such that

$$f(x) = y$$

i.e.,

$$a^x = y$$

As

$$a^x = y$$

On taking log both sides

$\Rightarrow$ $$x = \log_a y$$

$\therefore$ $$f(x) = y$$

Hence, f is onto.

$\therefore$ $$(\mathbb{R},+) \cong (\mathbb{R}_+,\cdot)$$

Let Q be the set of rational numbers and Q_+ be the set of positive rational number.

If f is homomorphism from Q to Q_+, then

$$f(x, y) = f(x)f(y) \ \forall \ x, y \in Q$$

And if image of 1 is known then the image of every element will be known.

$\therefore$ $f(x) = a^x$ where $a = f(1)$

If $a = 1$, $f(x) = 1$

$\therefore$ f is trivial homomorphism.

If $a \neq 1$, $f(x) = 1$

then $f(x) = a^x \in Q_+ \forall \ x \in Q$

which is a contradiction.

Hence, only trivial homomorphism is possible.

$\therefore$ $(Q, +) \not\cong (Q_+, .)$

1.2 Determine the number of homomorphisms from the additive group Z_{15} to the additive group Z_{10}. (Z_n is the cyclic group of order n).

(2009 : 12 Marks)

Solution:

Let $\phi : \mathbb{Z}_{15} \to \mathbb{Z}_{10}$ be a homomorphism.

As $\mathbb{Z}_{15}$ is a cyclic group of order 15.

$$\mathbb{Z}_{15} = \langle 1 \rangle$$

Under homomorphism, if element 1 will be mapped then remaining elements will get mapped themselves ($\because$ G is cyclic)

Suppose, $\phi(1) = x$

As we know, if f is homomorphism from G to G' then $O(f(a)/0(a)$ where $a \in G$.

As $\phi(1) = x$.

$\therefore$ $O(x)/O(a) = 15$

And order of element divides order of group

$\therefore$ $O(x)/10$ As $O(x)/15$

$O(x)/15 \Rightarrow O(x)|g.c.d(15, 10)O(x)|5$

$\therefore$ $O(x) = 1$ or $O(x) = 5$

If $O(x) = 1$. Then it is trivial homomorphism. And if $O(x) = 5$.

Note : In $\mathbb{Z}_n$, number of elements of order $k = \phi(k)$; provided $k|n$.

$\therefore$ In $\mathbb{Z}_{10}$, number of elements of order $5 = \phi(5) = 4$.

$\therefore$ We have 4 possibilities for x.

Total number of homomorphism = 4 + 1 = 5.

1.3 Show that the alternating group on four letters A_4 has no subgroup of order 6.

(2009 : 15 Marks)

Solution:

Consider the alternating group A_4.

$$\sigma(A_4) = \frac{\dot{\sigma}(S_4)}{2} = \frac{\underline{|4}}{2} = 12$$

We show although $6 | 12$, A_4 has no subgroup of order 6. Suppose H is a subgroup of A_4 and $\sigma(H) = 6$.

By previous problem the number of distinct 3-cycles in S_4 is

$$\frac{1}{3} \cdot \frac{4!}{(4-3)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 1} = 8$$

2 Real Analysis

1. Real Number System

1.1 Show that every open subset of R is a countable union of disjoint open intervals.

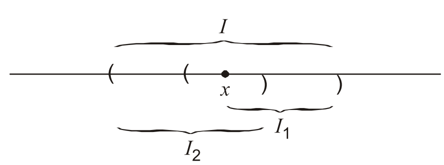
(2013 : 14 Marks)

Solution:

Let $U \leq R$ be an open and let $x \in U$. Then x is rational or irrational. If x is rational, define

$$I_x = \bigcup_{\substack{x \in I \subset U \\ I \text{ is open}}} I \quad \text{(Union of all open intervals in } U \text{ containing } x\text{)}$$

Note that I_x is simply the largest open interval containing x as it is union of disjoint open intervals (see figure below).



If x is irrational, as U is open, $\exists \in > 0$ such that $(x - \in, x + \in) \subset U$.

As every interval contains a rational $\exists y \in (x - \in, x + \in)$ and then $x \in I_y$.

$\therefore$ $\forall x \in U$ $x \in I_q$ for some $q \in U \cup Q$, i.e., where q is a rational in U.

$$\therefore \qquad U \subseteq \bigcup_{q \in U \cup Q} I_q$$

$$\text{But} \qquad I_q \leq U \,\forall q$$

$$\therefore \qquad U = \bigcup_{q \in U \cup Q} I_q$$

And as the number of rationals in U are countable U can be written as union of countable intervals. Also each of I_q is disjoint as it being maximal if two I_q overlap they will be same.

2. Sequences

2.1 Show that a bounded infinite subset of $\mathbb{R}$ must have a limit point.

(2009 : 15 Marks)

Solution:

Let A be an infinite and bounded set. There exist and interval $[k, K]$ such that $A \subseteq [k, K]$.
We define a set S as follows :
$x \in S$: It exceeds at the most a finite number of member of the set A.
Thus, while $k \in S$ and $K \in S$.
Also, S is bounded above in as much as K is an upper bounded of the same. Let ξ the least upper bound of S. Surely it exists by the order completeness properly of $\mathbb{R}$.

We show that ξ is a limit point of A.
Consider a neighbourhood say B of ξ,

$$\xi \in (a, b) \subset B$$

Now the number 'a' which is less than the least upper bound ξ of the set is not an upper bound of S. Thus there exist a number, η (sequence) of S such that

$$a < \eta \leq \xi, \eta \in S$$

Also η being a member of S exceeds at not a finite number of member of A : It follower and that the number 'a' also exceeds at the most a finite a finite number of member of A.
Again the number b which is greater than ξ is an upper bound of S without being a member of S. Thus must exceed an infinite number of member of A.
It follows that
(i) 'a' exceeds at the most a finite number of member of S.
(ii) 'b' exceeds an infinite number of members of S.
Thus, $[a, b]$ contains an infinite number of member of A. So, that ξ is the limit of point of S.

2.2 Discuss the convergence of the sequence $\{x_n\}$ where $x_n = \dfrac{\sin\frac{n\pi}{2}}{8}$.

(2010 : 12 Marks)

Solution:

Given : Sequence $\{x_n\} = \left\{\frac{1}{8}, 0, -\frac{1}{8}, 0, \frac{1}{8}, 0, -\frac{1}{8}, ...\right\}$

So, the given sequence $\{x_n\}$ assumes 3 values viz., 0, $-\frac{1}{8}$ and $\frac{1}{8}$ and is oscillatory in nature.

$\therefore$ $\{x_n\}$ does not converge.

2.3 Define $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$. Show that the sequence converges to $\dfrac{1+\sqrt{17}}{2}$.

(2010 : 12 Marks)

Solution:

Given : $x_1 = 5$ and $x_{n+1} = \sqrt{4+x_n}$ for $n > 1$

$$\therefore \quad x_2 = \sqrt{4+x_1} = \sqrt{4+5} = \sqrt{9} = 3$$

$$x_3 = \sqrt{4+x_2} = \sqrt{4+3} = \sqrt{7}$$

Let $n = 1$

$$\therefore \quad x_2 < x_1$$

$\therefore$ True for $n = 1$.
Let it is also true for $K \in N$.

$$\therefore \quad x_{k+1} < x_k$$

$$x_{k+1} + 4 < x_k + 4$$

$$\therefore \quad \sqrt{x_{k+1}+4} < \sqrt{x_k+4}$$

$$\Rightarrow \quad x_{k+2} < x_{k+1}$$

$\therefore$ True for $k + 1$ also.
So, by mathematical induction it is true for all $K \in N$.
$\therefore$ $\{x_n\}$ is monotonically decreasing sequence.

3 Complex Analysis

1. Analytic Functions

1.1 Let $f(z) = \dfrac{a_0 + a_1 z + \ldots.. + a_{n-1} z^{n-1}}{b_0 + b_1 z + \ldots.. + b_n z^n}$, $b_n \neq 0$. Assume that the zeroes of the denominator are simple.

Show that the sum of the residues of $f(z)$ at its poles is equal to $\dfrac{a_n - 1}{b_n}$.

(2009 : 12 Marks)

Solution:

Proof : As the zeroes of denominator are simple, so $b_0 + b_1 z + \ldots.. + b_n z^n$ can be written as

$$b_n (z - z_0)(z - z_1)(z - z_2) \ldots.. (z - z_{n-1})$$

or
$$f(z) = \frac{a_0 + a_1 z + \ldots.. a_{n-1} z^{n-1}}{b_n (z - z_0)(z - z_1)(z - z_2) \ldots.. (z - z_n - 1)} \qquad \ldots(1)$$

As maximum degree of 'z' in numerator is $n - 1$ and degree of denominator is n (i.e., degree of denominator is greater than degree of numerator).

So, $f(z)$ can be written as

$$f(z) = \frac{A_0}{z - z_0} + \frac{A_1}{z - z_1} + \frac{A_2}{z - z_2} + \ldots.. \frac{A_{n-1}}{z - z_{n-1}} \qquad \ldots(2)$$

$$= \frac{A_0 (z - z_1)(z - 2) \ldots.. (z - z_{n-1}) + A_1 (z - z_0)(z - z_2) - (z - z_{n-1}) + \ldots. A_n (z - z_0)(z - z_{n-2})}{(z - z_0)(z - z_1) \ldots.. (z - z_{n-1})} \qquad \ldots(3)$$

Now, we know that residues of $f(z)$ are $A_0, A_1, A_2, \ldots.., A_{n-1}$.

According to problem, we need to calculate $A_0 + A_1 + A_2 \ldots.. A_{n-1}$.

From (3), the coefficient of z^{n-1} in numerator will be calculated as :

$$A_0 + A_1 + A_2 \ldots.. A_{n-1}$$

because on observing

$$\begin{aligned} & A_0 (z - z_1)(z - z_2) - (z - z_{n-1}) \\ + \; & A_1 (z - z_0)(z - z_2) - (z - z_{n-1}) \\ + \; & A_2 (z - z_0)(z - z_1)(z - z_3) - (z - z_{n-1}) \\ + \; & \vdots \\ + \; & A_{n-1} (z - z_0)(z - z_1) \ldots.. (z - z_{n-2}) \end{aligned}$$

In each of terms respectively coefficient of z^{n-1} will be $A_0, A_1, A_2 \ldots.. A_{n-1}$. So, the overall coefficient of z^{n-1} in numerator will be $A_0 + A_1 + A_2 \ldots.. A_{n-1}$.

Now, comparing this coefficient of 2^{n-1} with (1) i.e.,

$$\frac{\dfrac{a_0}{b_n} + \dfrac{a_1}{b_n} z + \dfrac{a_2}{b_n} z^2 \ldots.. \dfrac{a_{n-1}}{b_n} z^{n-1}}{(z - z_0)(z - z_1) \ldots.. (z - z_{n-1})} \qquad \ldots(4)$$

In this fraction, coefficient of z^{n-1} will be $\frac{a_{n-1}}{b_n}$.

(as denominator $(z-z_0)(z-z_1)$ ..... $(z-z_{n-1})$ is common (3) and (4))

$$\therefore \quad A_0 + A_1 + A_2 \ldots. A_{n-1} = \frac{a_{n-1}}{b_n}$$

or $$\text{Sum of residues} = \frac{a_{n-1}}{b_n}.$$ **Thus Proved.**

1.2 Show that $u(x, y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of $u(x, y)$. Hence, find the analytic function f for which $u(x, y)$ is the real part.

(2010 : 12 Marks)

Solution:

Given $$u(x, y) = 2x - x^3 + 3xy^2$$

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = -6x$$

$$\frac{\partial u}{\partial y} = 6xy, \quad \frac{\partial^2 y}{\partial y^2} = 6x$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$$

$\therefore$ $u(x, y)$ is a harmonic function.

Let v be its harmonic conjugate.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2 - 3x^2 + 3y^2$$

$$\Rightarrow \quad v = 2y - 3x^2y + y^3 + f(x) \quad \ldots(1)$$

Also, $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 6xy$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = -6xy$$

$\Rightarrow$ $v = -3x^2y + g(y)$ ...(2) ($f(x)$ and $g(y)$ are some arbitrary functions)

Comparing (1) and (2), we get

$v = 2y + y^3 - 3x^2y + c$, where c is a constant.

Now, let $$f = u(x, y) + iv(x, y)$$

$$\Rightarrow \quad f = (2x - x^3 + 3xy^2) + i(2y + y^3 - 3x^2y + c)$$

$$f(z) = \int (\phi_1(z,0) - i\phi_2(z,0))dz$$

$$= \int (2 - 3z^2)dz = 2z - z^3 + A$$, where A is a constant

$$\Rightarrow \quad f(z) = 2z - z^3 + A$$

1.3 (i) Evaluate the line integral $\int_c f(z)dz$ where $f(z) = z^2$, c is the boundary of the triangle with vertices $A(0, 0)$, $B(1, 0)$, $C(1, 2)$ in that order.

(ii) Find the image of the finite vertical strip $R : x = 5$ to $x = 9$, $-\pi \le y \le \pi$ of z-plane under exponential function.

(2010 : 15 Marks)

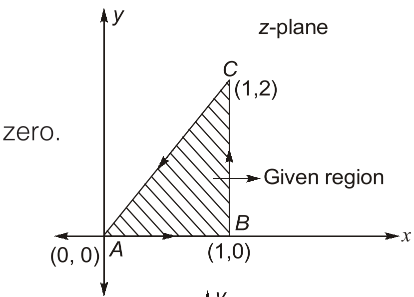
Solution:

(i) Given : $f(z) = z^2$

Now, $f(z) = z^2$ is analytic in whole given region.

Line integral of cled boundary curve of analytic function is always zero.

$\therefore \int_c f(z)dz$ in the given region is 0.



(ii) Given : x varies from 5 to 9 and y varies from $-\pi$ to π.

Now, $z = x + iy$

Image under exponential function will be

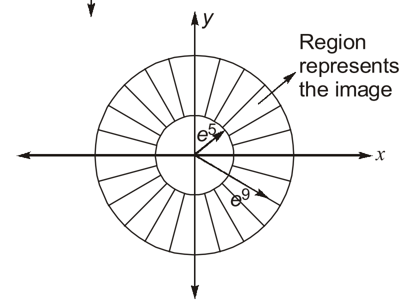
$$e^z = e^{x+iy} = e^x.e^{iy}$$

As x varies from 5 to 9, $\therefore$ e^x varies from e^5 to e^9.

Also, e^{i4} denotes angle variation.

So, image is given by region in figure.

So, the region is bounded by two circles of radii e^5 and e^9.



1.4 If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition, $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$.

(2011 : 12 Marks)

Solution:

Let $f(z) = u + iv$

$\Rightarrow$ $if(z) = iu - v$

$\Rightarrow$ $(1+i)f(z) = u - v + i(u + v)$

$= u + iv$

$$\therefore \quad U = u - v = \frac{e^y - \cos x + \sin x}{\cos hy - \cos x}$$

$$= \frac{\cos hy + \sin hy - \cos x + \sin x}{\cos hy - \cos x}$$

$$= 1 + \frac{\sin hy + \sin x}{\cos hy - \cos x}$$

Let $\dfrac{\partial U}{\partial x} = \phi_1(x, y)$ and $\dfrac{\partial U}{\partial y} = \phi_2(x, y)$

$$\therefore \quad \phi_1(x, y) = \frac{\partial U}{\partial x} = \frac{\cos x(\cos hy - \cos x) - \sin x(\sin hy + \sin x)}{(\cos hy - \cos x)^2}$$

$$\therefore \quad \phi_1(z, 0) = \frac{\cos z(1 - \cos z) - \sin^2 z}{(1 - \cos z)^2} = \frac{\cos z - 1}{(1 - \cos z)^2}$$

$$= \frac{-1}{1 - \cos z} = \frac{-1}{2}\operatorname{cosec}^2 \frac{z}{2}$$

$$\phi_2(x, y) = \frac{\partial U}{\partial y} = \frac{\cos hy(\cos hy - \cos x) - \sin hy(\sin hy + \sin x)}{(\cos hy - \cos x)^2}$$

$$\therefore \quad \phi_2(z, 0) = \frac{1 - \cos z}{(1 - \cos z)^2} = \frac{1}{1 - \cos z}$$

4 Linear Programming Problems

1. Basic Feasible Solutions

1.1 By the method of Vogel, determine an initial basic feasible solution for the following transportation problem :

Product P_1, P_2, P_3 and P_4 have to be sent to destinations D_1, D_2 and D_3. The cost of sending product P_i to destinations D_j is C_{ij}, where the matrix

$$[C_{ij}] = \begin{bmatrix} 10 & 0 & 15 & 5 \\ 7 & 3 & 6 & 15 \\ 0 & 11 & 9 & 13 \end{bmatrix}$$

The total requirements of destinations D_1, D_2 and D_3 are given by 45, 45, 95 respectively and the availability of the products P_1, P_2, P_3 and P_4 are respectively 25, 35, 55 and 70.

(2012 : 20 Marks)

Solution:

The above problem can be expressed as

	D_1	D_2	D_3	Supply		
P_1	[10]	[7]	[0] 25	(25)	(7)	(3)
P_2	[0]	[3] 20	[11] 15	(35)	(3)	(3)
P_3	[15]	[6]	[9] 55	(55)	(3)	(9)
P_4	[5] 45	[15] 25	[13]	(70)	(8)	(10)
Demand	(45)	(45)	(95)			
	(5)	(3)	(9)			

∴ By Vogel's method, an initial basic feasible solution of the given problem is given by the above table.

$$\text{Total transportation cost} = 3 \times 20 + 11 \times 15 + 9 \times 55 + 5 \times 45 + 15 \times 25$$
$$= 60 + 165 + 495 + 225 + 375 = \text{Rs. } 1320$$

Explanation of Vogel's Method :

1. The given values C_{ij} (given cost) are written in the upper left corner.
2. Find the difference between the smallest and next to the smallest C_{ij}'s across rows and columns and write at the bottom and extreme right of the boxes.
3. Select the row/column having the largest difference.
4. Allocate the maximum possible to the smallest cost and write in the middle.
5. Cross-out the columns/rows whose cost/demand gets completely filled.

1.2 How many basic solutions are there in following linearly independent set of equations? Find all of them.

$$2x_1 - x_2 + 3x_3 + x_4 = 6$$
$$4x_1 - 2x_2 - x_3 + 2x_4 = 10$$

(2018 : 15 Marks)

Solution:

Form the table from given set of equations :

Basic Variables	Non - Basic Variables	Solution	Is it possible?
x_1, x_2	$x_3 = x_4 = 0$	Inconsistent	No
x_1, x_3	$x_2 = x_1 = 0$	$x_1 = 2.57$ $x_3 = 0.236$	Yes
x_1, x_4	$x_2 = x_3 = 0$	Inconsistent equation	No
x_2, x_3	$x_1 = x_4 = 0$	$x_2 = -5.14$ $x_3 = 0.286$	Yes
x_2, x_4	$x_1 = x_3 = 0$	Inconsistent equation	No
x_3, x_4	$x_1 = x_2 = 0$	$x_3 = 0.286$ $x_1 = 5.14$	Yes

∴ There are 3 basic solutions.

(i) $x_1 = 2.57, x_2 = 0, x_3 = 0.286, x_4 = 0$

(ii) $x_1 = 0, x_2 = -5.14, x_3 = 0.286, x_4 = 0$

(iii) $x_1 = 0, x_2 = 0, x_3 = 0.286, x_4 = 5.14$

1.3 UPSC maintenance section has purchased sufficient number of curtain cloth pieces to meet the curtain requirement of its building. The length of each piece is 17 feet. The requirement according to curtain length is as follows:

Curtain length (in feet)	Number required
5	700
9	400
7	300

The width of all curtains is same as that of available pieces. Form a linear programming problem in standard form that decides the number of pieces cut in different ways so that the total trim loss is minimum. Also give a basic feasible solution to it.

(2020 : 10 Marks)

Solution:

Given, Length of curtain = 17 ft

Quantity	Curtain length 9	7	5	Loss (ft)
x_1	1	1	0	1
x_2	1	0	1	3
x_3	0	1	2	0
x_4	0	2	0	3
x_5	0	0	3	2

To minimize,

$$z = x_1 + 3x_2 + 0x_3 + 3x_4 + 2x_5$$
$$= x_1 + 3x_2 + 3x_4 + 2x_5$$

For 5 feet curtain,

$$0x_1 + x_2 + 2x_3 + 0x_4 + 3x_5 \geq 700 \quad \text{...(i)}$$

5 Partial Differential Equations

1. Formulation of P.D.E.

1.1 Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that this equation is satisfied by the surface $yz + zx + xy = 0$.

(2009 : 12 Marks)

Solution:

The equation cone having vertex at origin

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \quad \text{...(i)}$$

where a, b, c, f, g, h are parameters.

Differentiating w.r.t. x and y, we get

$$2ax + 2hy + 2gz + 2gxp + 2zcp + 2fyp = 0$$

And $$2by + 2czq + 2hx + 2fyq + 2fz + 2gxq = 0$$

So, $$ax + hy + qz + p(gx + zc + fy) = 0 \times x$$

$$by + hx + fz + q(cz + fy + gx) = 0 \times y$$

$\Rightarrow$ $$ax^2 + hxy + gzx + p(gx^2 + czx + fyx) = 0$$

$$by^2 + hxy + fzy + q(czy + fy^2 + gxy) = 0$$

On adding,

$\Rightarrow$ $$ax^2 + by^2 + 2hxy + gzx + fzy + px + qy[cz + fy + gx] = 0$$

$\Rightarrow$ $$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0$$

$\Rightarrow$ $$(cz + fy + gx)(px + qy - z) = 0$$

Clearly, $px + qy - z = 0$ is required differential equation.

Given surface is $yz + zx + xy = 0$ (*)

Differentiating (*) w.r.t. x and y, we get

$$yp + z + px + y = 0 \quad \text{...(2)}$$

$$z + yq + xq + x = 0 \quad \text{...(3)}$$

So, we get $$p = \frac{-(z+y)}{(x+y)}, q = \frac{-(x+z)}{(x+y)}$$

$$px + qy - z = \frac{-(z+y)x}{(x+y)} - \frac{(x+z)}{(x+y)}y - z$$

$$= \frac{-(z+y)x - (x+z)y - z(x+y)}{(x+y)}$$

$$= \frac{-xz - xy - xy - zy - zx - zy}{(x+y)}$$

$$= \frac{-2(xy + yz + zx)}{x+y} = \frac{-20}{x+y} = 0$$

1.2 From the partial differential equation by eliminating the arbitrary function f given by :

$$f(x^2 + y^2, z - xy) = 0$$

(2009 : 6 Marks)

Solution:

The function is

$$z = xy + F(x^2 + y^2) \quad \text{...(1)}$$

Now differentiating partially (1) w.r.t. x we get

$$\frac{\partial z}{\partial x} = y + F'(x^2 + y^2)2x$$

So, $$\frac{p - y}{2x} = F'(x^2 + y^2) \quad \text{...(2)}$$

Now, differentiating partially (1) w.r.t. y, we get

$$\frac{\partial z}{\partial y} = x + F'(x^2 + y^2).2y$$

So, $$\frac{q - x}{2y} = F'(x^2 + y^2) \quad \text{...(3)}$$

Equating (2) and (3), we get

$$\frac{p - y}{2x} = \frac{q - x}{2y}$$

So, $py - qx = y^2 - x^2$ is linear PDE.

1.3 Find the surface satisfying the P.D.E. $(D^2 - 2DD' + D'^2)z = 0$ and the conditions that $bz = y^2$ when $x = 0$ and $az = x^2$ when $y = 0$.

(2010 : 12 Marks)

Solution:

Given, the equation is

$$(D^2 - 2DD' + D'^2)z = 0$$

$$\Rightarrow \quad (D - D')^2 z = 0$$

The auxiliary eqn. for above eqn. is

$$(m - 1)^2 = 0$$

$$\Rightarrow \quad m = 1, 1$$

$\therefore$ The solution of above eqn. is

$$z = \phi_1(y + x) + x\phi_2(y + x)$$

Given, at $x = 0$, $bz = y^2 \Rightarrow$ $$z = \frac{y^2}{b}$$

i.e., $$\frac{y^2}{b} = \phi_1(y) + 0 \Rightarrow \phi_1(y) = \frac{y^2}{b}$$

$$\Rightarrow \quad \phi_1(x + y) = \frac{(y + x)^2}{b}$$

at $y = 0$, $az = x^2$ $\Rightarrow$ $$z = \frac{x^2}{a}$$

i.e., $$\frac{x^2}{a} = \phi_1(x) + x\phi_2(x)$$

$$\Rightarrow \quad \frac{x^2}{a} = \frac{x^2}{b} + x\phi_2(x)$$

$$\Rightarrow \quad x\phi_2(x) = x^2\left(\frac{1}{a} - \frac{1}{b}\right) \Rightarrow \phi_2(x) = x\left(\frac{1}{a} - \frac{1}{b}\right)$$

6 Numerical Analysis and Computer Programming

1. Solution of Algebraic and Transcendental Equation of One Variable

1.1 Develop an algorithm for Regula-Falsi method to find a root of $f(x) = 0$ starting with two initial iterates x_0 and x_1 to the root such that sign $(f(x_0)) \neq \sin(f(x_1))$. Take n as the maximum number of iterations allowed and eps be the prescribed error.

(2009 : 30 Marks)

Solution:

1. Read x_0, x_1, eps, n

   Remarks : x_0 and x_1 are two initial guesses to the root such that $\sin(f(x_0)) \neq$ sign $(f(x_1))$. The prescribed precision is eps and n is the maximum number of iterations. Steps 2 and 3 are initialization steps.
2. $f_0 \leftarrow x\ f(x_0)$
3. $f_1 \leftarrow f(x_1)$
4. For $i = 1$ to n in steps of 1 do.
5. $x_2 \leftarrow (x_0 f_1 - x_1 f_0)/(f_1 - f_0)$
6. $f_2 \leftarrow f(x_2)$
7. if $|f_2| \leq$ eps then
8. begin write 'convergent solution', x_2, f_2
9. stop end
10. if sign $(f_2) \neq$ sign (f_0)
11. Then begin $x_1 \leftarrow x_2$
12. $f_1 \leftarrow f_2$ end
13. else begin $x_0 \leftarrow x_2$
14. $f_0 \leftarrow f_2$ end

    end for
15. Write 'Does not converge in n iterations'.
16. Write x_2, f_2
17. Stop

1.2 Find the positive root of the equation

$$10xe^{-x^2} - 1 = 0$$

correct upto 6 decimal places by using Newton-Raphson method. Carry out computations only for three iterations.

(2010 : 12 Marks)

Solution:

Given : $$f(x) = 10xe^{-x^2} - 1$$

$\therefore$ $$f'(x) = 10e^{-x^2} + 10xe^{-x^2} \times (-2x)$$

$$= 10e^{-x^2}(1 - 2x^2)$$

Let $x_0 = 0$

Using Newton-Raphson's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - (-0.1) = 0.1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.101026$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.101026$$

$\therefore$ The positive root is 0.101026 (approx.)

1.3 Use Newton-Raphson method to find the real root of the equation $3x = \cos x + 1$ correct to four decimal places.

(2012 : 12 Marks)

Solution:

The given equation is

$$f(x) = 3x - \cos x - 1 = 0$$

$f(0.60) < 0$ and $f(0.61) > 0$

Hence, a real root of $f(x)$ lies in the interval (0.60, 0.61)

$$f'(x) = 3 + \sin x$$

From Newton-Raphson's method, we have

$$x_{x+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

Taking $x_0 = 0.60$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{3x_0 - \cos x_0 - 1}{3 + \sin x_0}$$

$$= 0.60 - \frac{3(0.60) - \cos(0.60) - 1}{3 + \sin(0.60)}$$

$$= 0.60701$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= x_1 - \frac{3x_1 - \cos x_1 - 1}{3 + \sin x_1}$$

$$= 0.60701 - \frac{3(0.60701) - \cos(0.60701) - 1}{3 + \sin(0.60701)}$$

$$= 0.60710$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= x_2 - \frac{3x^2 - \cos x_2 - 1}{3 + \sin x_2}$$

$$= 0.60710 - \frac{3(0.60710) - \cos(0.60710) - 1}{3 + \sin(0.60710)}$$

$$= 0.60710$$

Since $x_2 = x_3$, $\therefore$ $x = 0.60710$ is a root of $f(x)$.

1.4 Apply Newton-Raphson method to determine a root of the equation $\cos x - xe^x = 0$. Correct upto four decimal places.

(2014 : 10 Marks)

Solution:

$$f(x) = \cos x - xe^x$$

So that, $f(0) = 1$

and $f(s) = \cos 1 - e = -2.17798$

$\therefore$ $f(0)f(i) < 0$

Hence the root lies between 0 and 1.

Take $x_0 = 0$ and $x_1 = 1$

$\therefore$ $f(x, 0) = 1$ and $f(x_1) = -2.17798$

By the method of fase position, we get

$$x_2 = \frac{x_0 f(x_1) - x_1(x_0)}{f(x_1) - f(x_0)} \quad \text{. . .(i)}$$

$$= \frac{0(-2.17798) - 1(1)}{-2.1779.8 - 1} = 0.31467$$

$\therefore$ The first approximation to the root is

$$x_2 = 0.31467$$

Now $f(x_2) = 0.51987 > 0$

$f(x_2)f(x_1) < 0$

$\therefore$ The root lies between 0.31467 and 1.

Take $x_0 = 0.31467$ and $x_1 = 1$

$\therefore$ $f(x_0) = 0.51987$ and $f(x_1) = -2.17798$

From (v), $$x_3 = \frac{(0.3146)(-2.17798) - 1(0.51987)}{-2.17798 - 0.5187} = 0.44673$$

The 2nd approximation to the root is,

$$x_3 = 0.44673$$

Now repeating this process, the successive approximations are

$$x_4 = 0.49402,\ x_5 = 0.50995$$

$$x_6 = 0.51520,\ x_7 = 0.51692,\ x_8 = 0.51748$$

$$x_9 = 0.51767,\ x_{10} = 0.51775 \text{ etc.}$$

$\therefore$ The approximate root is 0.5177 correct to 4 decimal places.

1.5 Apply Newton-Raphson method, to find a real root of transcendental equation $x \log_{10} x = 1.2$, correct to three decimal places.

(2019 : 10 Marks)

Solution:

Here, $x \log x = 1.2$

i.e., $x \log x - 1.2 = 0$

7 Mechanics and Fluid Dynamics

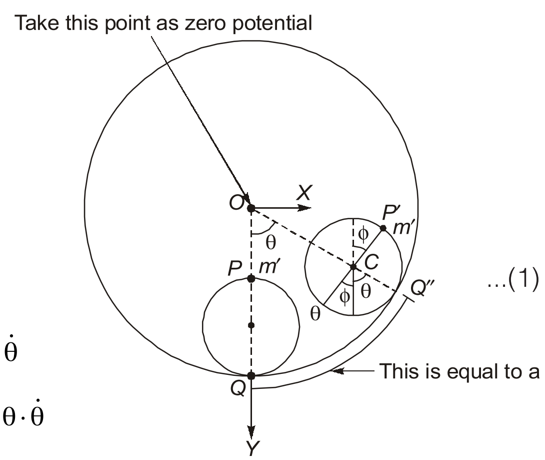
1. Generalised Coordinate

1.1 A perfectly rough sphere of mass m and radius b, rests on the lowest point of a fixed spherical cavity of radius a. To the highest point of the movable sphere is attached a particle of mass m' and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length

$$(a-b)\frac{4m'+\frac{7}{5}m}{m+m'\left(2-\frac{a}{b}\right)}.$$

(2009 : 30 Marks)

Solution:



Now, $QQ'' = Q'Q''$

or $a\theta = b(\theta+\phi)$

$$(a-b)\theta = b\phi$$

$$(a-b)\dot{\theta} = b\dot{\phi}$$

Now, total kinetic energy, T can be written as

$$T = \frac{1}{2}m'V_p^2 + \frac{1}{2}mV_{cm}^2 + \frac{1}{2}I_{cm}w^2 \qquad \text{...(1)}$$

Now, $x_c = (a-b)\sin\theta \Rightarrow \dot{x}_c = (a-b)\cos\theta\cdot\dot{\theta}$

$$y_c = (a-b)\cos\theta \Rightarrow \dot{y}_c = (a-b)-\sin\theta\cdot\dot{\theta}$$

$$v_c^2 = \dot{x}_c^2+\dot{y}_c^2 = (a-b)^2\dot{\theta}^2$$

$$I_{cm} = \frac{2}{5}ma^2 \text{ and } w = \dot{\phi}$$

$$x_p = (a-b)\sin\theta + b\sin\phi$$

$$\dot{x}_p = (a-b)\cos\theta\cdot\dot{\theta} + b\cos\phi\cdot\dot{\phi}$$

$$y_p = (a-b)\cos\theta - b\cos\phi$$

$$\dot{y}_p = (a-b)(-\sin\theta)\cdot\dot{\theta} + b\sin\phi\cdot\dot{\phi}$$

$$\dot{x}_p^2+\dot{y}_p^2 = v_p^2 = (a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2(a-b)\dot{\theta}\dot{\phi}b(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$= (a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2(a-b)\dot{\theta}\dot{\phi}(1\cdot 1-\theta\phi) \quad \text{(Ignoring the term } \theta\phi \text{ as it is very small)}$$

$$\approx (a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2(a-b)\dot{\theta}\dot{\phi}$$

Now potential energy, $V = -mg(a-b)\cos\theta - m'g\{(a-b)\cos\theta - b\cos\phi\}$

Now lagrange, $L = T - V$

$$L = \frac{1}{2}m'\{(a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2(a-b)b\dot{\theta}\dot{\phi}\} +$$

$$\frac{1}{2}m(a-b)^2\dot{\theta}^2 + \frac{1}{2}\times\frac{2}{5}ma^2\dot{\phi}^2 + mg(a-b)\cos\theta +$$
$$m'g\{(a-b)\cos\theta - b\cos\phi\}$$

As $(a-b)\theta = b\phi$

So, the generalised coordinate will only be one.

Convert θ to ϕ.

Also $(a-b)\dot{\theta} = b\dot{\phi}$

So,
$$L = \frac{1}{2}m'\{b\dot{\phi}^2 + b^2\dot{\phi}^2 + 2b^2\dot{\phi}^2\} + \frac{1}{2}mb^2\dot{\phi}^2 + \frac{ma^2}{5}\dot{\phi}^2 + mg(a-b)\cos\theta$$
$$+ m'g(a-b)\cos\theta - m'gb\cos\phi$$

$$L = \dot{\phi}^2\left(2m' + \frac{7m}{10}\right)b^2 + (m+m')g(a-b)\cos\frac{b\phi}{a-b} - m'gb\cos\phi$$

$$\frac{\partial L}{\partial\dot{\phi}} = 2\dot{\phi}\left(2m' + \frac{7m}{10}\right)b^2$$

$$\frac{\partial L}{\partial\phi} = \frac{b}{(a-b)}(m+m')g(a-b)(-1)\sin\frac{b\phi}{(a-b)} + m'gb\sin\phi$$
$$= -b(m+m')g\sin\frac{b\phi}{(a-b)} + m'gb\sin\phi$$

But $\sin\frac{b\phi}{a-b} \approx \frac{b\phi}{a-b}$ and $\sin\phi \approx \phi$

So,
$$\frac{\partial L}{\partial\phi} = -b(m+m')g\frac{b\phi}{(a-b)} + m'bg\phi$$

Now, using Lagrange's theorem

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

or
$$\ddot{\phi}\left(4m' + \frac{7m}{5}\right)b^2 + b(m+m')\frac{gb\phi}{a-b} - bm'g\phi$$

$$\ddot{\phi}\left(4m' + \frac{7m}{5}\right)b^2 + \phi bg\left[\frac{b(m+m') - m'(a-b)}{a-b}\right] = 0$$

$$\ddot{\phi} + \frac{gb}{\left(4m' + \frac{7m}{5}\right)b^2}\left(\frac{m'(2b-a) + bm}{(a-b)}\right)\phi = 0$$

This is S.H.M. equation.

or $$\ddot{\phi} + w^2\phi = 0$$

So,
$$w^2 = \frac{g\left\{m'\left(2 - \frac{a}{b}\right) + m\right\}}{\left(4m' + \frac{7m}{5}\right)(a-b)} = \frac{g}{L}$$

or,
$$L = \frac{\left(4m' + \frac{7m}{5}\right)(a-b)}{m + m'\left(2 - \frac{a}{b}\right)}$$

So, oscillations are the same as those of a simple pendulum of above length

$$L = \frac{\left(4m' + \frac{7m}{5}\right)(a-b)}{m + m'\left(2 - \frac{a}{b}\right)}$$

1.2 A sphere of radius *a* and mass *m* rolls down a rough plane inclined at an angle α to the horizontal. If *x* be the distance of the point of contact of sphere from a fixed point on the plane, find the acceleration by using Hamilton's equations.

(2010 : 30 Marks)

Solution:

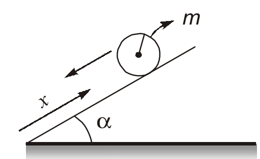


Figure-1

Figure-1 below depicts the situation given in problem.

$K \rightarrow$ Radius of gyration of given sphere

Kinetic energy, $T = \frac{1}{2}m(\dot{x}^2 + K^2\dot{\theta}^2)$

$= \frac{1}{2}m\left(\dot{x}^2 + \frac{2}{5}a^2\dot{\theta}^2\right)$ $\left(K = \frac{2}{a}a^2 \text{ for sphere}\right)$

$= \frac{1}{2}m\left(\dot{x}^2 + \frac{2}{5}\dot{x}^2\right) = \frac{7}{10}m\dot{x}^2$ (In pure rolling, $\dot{x} = a\dot{\theta}$)

Potential energy, $V = -mfx\sin$

$\therefore$ $L = T - V = \frac{7}{10}m\dot{x}^2 + mfx\sin\alpha$

Now, $P_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{10}m \times 2\dot{x} = \frac{7}{5}m\dot{x}$

$\Rightarrow$ $\dot{x} = \frac{5P_x}{7m}$

$\therefore$ Hamiltonian, $H = -L + P_x \cdot \dot{x} = -\frac{7}{10}m\dot{x}^2 - mgx\sin\alpha + P_x \cdot \frac{5P_x}{7m}$

$\Rightarrow$ $H = \frac{-5P_x^2}{14m} - mgx\sin\alpha + \frac{5P_x^2}{7m}$ $\left(\text{Putting } \dot{x} = \frac{5P_x}{7m}\right)$

$\Rightarrow$ $H = \frac{5}{14m}P_x^2 - mgx\sin\alpha$

$\therefore$ One of the Hamiltonian's equation gives

$$\dot{P}_x = \frac{-\partial x}{\partial x} = +mf\sin\alpha$$

$\Rightarrow$ $\frac{7}{5}m\ddot{x} = xf\sin\alpha$

$\Rightarrow$ $\ddot{x} = \frac{5}{7}g\sin\alpha$

$\therefore$ Acceleration of sphere is $\frac{5}{7}g\sin\alpha$.

1.3 Obtain the equations governing the motion of a spherical pendulum.

(2012 : 12 Marks)

Solution:

Let '*m*' be the mass of the bob of spherical pendulum, which can swing in any direction, traces out a sphere of constant length *l*.