A Text Book on

Industrial Engineering, Robotics and Mechatronics

Useful for IAS / GATE / ESE / PSUs and other competitive examinations

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MADE EASY Publications
A text book on Industrial Engineering, Robotics and Mechatronics

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FOREWORD

In my long teaching career I have come across many other fellow teachers but Dr. Swadesh Singh is unique. He has the rare combination of being Ex-IES officer in Govt. of India, a Ph.D from reputed Indian Institute of Technology Delhi, a Young Scientist award winner and a Career Award winner for teaching. As a teacher, he is one of those who really tries to understand the students from their perspective and helps them. He has a penchant for guiding and counseling students in choosing their careers and preparing them for their competitive exams. Many students have benefited from him and holds regard for him even years after graduating. Dr. Singh’s book on Production Technology has become popular and is a standard among the students who are top rankers in competitive exams like GATE, CIVIL SERVICES etc. and the present book on Industrial engineering should stand testimony to the expertise of Dr. Singh’s in presenting in a compact book what is just appropriate for preparing for such exams.

This book is a must-have for those who are serious in finding success in competitive exams.

Prof. P. S. Raju  
Director  
GRIET, Hyderabad, India
PREFACE

After the overwhelming response of my first book on Production Engineering, I thought of writing another book on the core subject of Industrial Engineering. After teaching for various competitive examinations for about 16 years, I found that although there are so many books available on the fundamentals of Industrial Engineering but there is no book available for competitive examinations. The book which student can have and crack any question that appears in examination like GATE, IES and other public sector examination. So in the present book my effort is to teach fundamentals by problem solving so that students can understand well. Apart from the solved example-problems, numerous objective-type practice problems from various competitive examinations are given, along with the answer keys. The first edition of this book I released in 2011. Since most of the work including the typesetting, editing and so on was carried out by me, there were many grammatical mistakes in that edition. While teaching and students solving the problem we have corrected the text thoroughly and I have also added some more relevant text in each topic making it more suitable for the competitive examinations. Along with Dr. Rajesh Purohit of MANIT Bhopal, I have added Robotics and Mechatronics to make the book suitable for IES and other competitive examinations.

I express my gratitude to my spiritual teacher who has given me inspiration to write this book. Every subject matter can be learned through the medium of a teacher only. He thought me both his precept and by his personal example how to be sincere at work and what is good for me and how can I help others in a genuine way.

The authors are thankful to Sunil Kumar, Deepen Banorinya, Utkarsh Pandey, Bhrant Kumar, Harish, Limbadri, Prudvi, Gangadhar and Srinivasu for taking pain in correcting the manuscript many times. We also thank MADE EASY proof editing staff and CMD Sri B. Singh for their full support in publishing this book.

We hope that aspirants of scoring high in GATE, ESE, IAS, PSU and any competitive exams find this book as a helpful aid in their preparations.

Dr. Swadesh Kumar Singh
Dr. Rajesh Purohit
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Graphical Method

1.1 INTRODUCTION

Optimization is a process of either maximizing or minimizing a specific quantity called objective which depends upon a finite number of variables. These variables can be altogether independent or related to each other. Depending upon the availability of facilities in the department or plant, constraints will be formed. Graphical Method of solving these problems is limited to two dimensions only and the method can be understood by the following examples.

Example: 1.1

Assume that the following specify a generalized linear programming problem:

\[ \text{Max } Z = 45x + 40y \]

Subject to

\[ \begin{align*}
2x + y & \leq 90 \\
x + 2y & \leq 80 \\
x + y & \leq 50 \\
x, y & \geq 0
\end{align*} \]

Solve it by using Graphical Method and slope method.

Solution:

(a) The equation of lines can be changed into this form

\[ \begin{align*}
2x + y & \leq 90 \Rightarrow \frac{x}{45} + \frac{y}{90} \leq 1 \\
x + 2y & \leq 80 \Rightarrow \frac{x}{80} + \frac{y}{40} \leq 1 \\
x + y & \leq 50 \Rightarrow \frac{x}{50} + \frac{y}{50} \leq 1
\end{align*} \]

(b) Now plot these lines on the graph.

(c) Mark the feasible region as shown in above graph.

(d) Locate the end points of feasible region as O, A, B, C, D

(e) Find ‘x’ and ‘y’ coordinates of those points as shown below:

\[ \begin{align*}
Z_{\text{at } O} &= 45(0) + 40(0) = 0 \\
Z_{\text{at } A} &= 45(0) + 40(40) = 1600 \\
Z_{\text{at } B} &= 45(20) + 40(30) = 2100 \\
Z_{\text{at } C} &= 45(40) + 40(10) = 2200 \rightarrow \text{optimum point} \\
Z_{\text{at } D} &= 45(45) + 40(0) = 2025
\end{align*} \]
(f) To find point of maxima directly we need to understand the following theory:

- Initially the objective function line is drawn passing through the origin, the moment the objective function line first touches the feasible region or feasible point that is the point of minima.
- The farthest point at which objective function line is touching the feasible region is the point of maxima.

(g) % Utilization of machines at the optimum point:
Now optimum point is (40,10) so,
Utilization of machine 1 is

\[ 2x + y = 2(40) + 10 = 90 \leq 90 \]

\[ \therefore \text{% Utilization of machine 1 is } \frac{90}{90} \times 100 = 100\% \]

Utilization of machine 2 is
\[ x + 2y = (40) + 2(10) = 60 \leq 80 \]

\[ \therefore \text{% Utilization of machine 2 is } \frac{60}{80} \times 100 = 75\% \]

Utilization of machine 3 is
\[ x + y = 40 + 10 = 50 \leq 50 \]

\[ \therefore \text{% Utilization of machine 3 is } \frac{50}{50} \times 100 = 100\% \]

Slope Method:

- Slope of the objective function \( m = \frac{-45}{40} \)
- Slope of first constraint \( m_1 = -2 \)
- Slope of second constraint \( m_2 = \frac{1}{2} \)
- Slope of third constraint \( m_3 = -1 \)

As \( m \) lies between \( m_2 \) and \( m_1 \), the optimum point will be the intersection of line (1) and (3).

Note: If \( m_3 < m_2 < m_1 \) and
If \( m \in (m_1, m_2) \) then optimum point will be the intersection of line (1) with y axis.
But if \( m \in (0, m_3) \) then optimum point will be the intersection of line (3) with x axis.
(\text{It should be noted that in certain cases this method may not give the right answer because in certain cases the point may not appear in the feasible region).}

Example: 1.2

A certain company produces tea trays and ash trays out of sheet metal. Following data is given on capacity availability and economics of each product:

<table>
<thead>
<tr>
<th>Department</th>
<th>Time Taken for Ash Tray</th>
<th>Time Taken for Tea Tray</th>
<th>Total time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stamping</td>
<td>10 sec</td>
<td>20 sec</td>
<td>30000 sec</td>
</tr>
<tr>
<td>Forming</td>
<td>15 sec</td>
<td>5 sec</td>
<td>30000 sec</td>
</tr>
</tbody>
</table>

Determine the optimum production schedule. Each ash tray contributes ₹ 20 to the gross profit and each tea tray contributes ₹ 30 to the gross profit. Total daily fixed costs amount to ₹ 45,000. Solve this problem graphically what is the maximum net profit per day at the optimum production level, including the effect of fixed cost.
Solution:
Let number of ash trays = \( x \rightarrow \) Profit (₹ 20)
Number of tea trays = \( y \rightarrow \) Profit (₹ 30)
Fixed cost = 45,000
\[ \therefore \text{Objective function (max. } Z \text{) } = 20x + 30y - 45,000 \]

For stamping machine \( \Rightarrow 10x + 20y \leq 30,000 \Rightarrow \frac{x}{3000} + \frac{y}{1500} \leq 1 \)
For forming machine \( \Rightarrow 15x + 5y \leq 30,000 \Rightarrow \frac{x}{2000} + \frac{y}{6000} \leq 1 \)

\( O \ (0, \ 0) \Rightarrow Z_{O} = -45,000 \)
\( A(2000, \ 0) \Rightarrow Z_{A} = 5,000 \)
\( B(1800, \ 600) \Rightarrow Z_{B} = 9,000 \)
\( C(0, \ 1500) \Rightarrow Z_{C} = 0 \)

\( \therefore \text{ ₹ 9000 is the net profit per day} \)

Example: 1.3
Assume that the following specify a generalized linear programming problem:
\[ \text{Max } f(x) = 2x_{1} + x_{2} \]

Subject to \( x_{1} + x_{2} \leq 6 \)
\( x_{1} \leq 3 \)
\( 2x_{1} + x_{2} \geq 4 \)
\( x_{1}, \ x_{2} \geq 0 \)

Graph this problem, identifying the three constraint equation lines and the feasible zone common to all of them. Plot dotted lines for values of 3, 6, 9 and 12 for the objective function \( f(x) \). What appears to be the highest feasible value of \( f(x) \) and for what values of \( x_{1} \) and \( x_{2} \) does it occur?

Solution:
\[ x_{1} + x_{2} \leq 6 \Rightarrow \frac{x_{1} + x_{2}}{6} \leq 1 \]
\[ x_{1} \leq 3 \Rightarrow \frac{x_{1}}{3} \leq 1 \]
\[ 2x_{1} + x_{2} \geq 4 \Rightarrow \frac{x_{1} + x_{2}}{2} \geq 1 \]
\[ x_{1}, \ x_{2} \geq 0 \]

Corresponding to \( Z = 12 \), there is over utilization of all facilities
Corresponding to \( Z = 9 \), since it is an optimum value there is 100% utilization of all facilities.
At \( Z = 6 \), there is under utilization of all facilities.
Corresponding to \( Z = 3 \), it is not a feasible solution.
By slope method
\[ Z = 2x_{1} + x_{2} \]
\[ m = -2 \]
\[ x_1 + x_2 \leq 6 \quad m_1 = -1 \]
\[ x_1 \leq 3 \quad m_2 = \infty \]
\[ 2x_1 + x_2 \geq 4 \quad m_3 = -2 \]

Here \( m \) and \( m_3 \) are equal but it is important to note that, in maximization of \( Z \) value, we will consider the equations which always have less than or equal to “\( \leq \)” sign, and not consider the equation having “\( \geq \)” sign. In minimization, the converse is true.

**Example: 1.4**

Assume that the following specify a generalized linear programming problem:

\[ \text{Min } Z = 2x + 3y \]

Subjected to

\[ 5x + y \geq 100 \]
\[ 3x + 2y \geq 120 \]
\[ x + 3y \geq 90 \]
\[ x, y \geq 0 \]

Solve it by using Graphical Method.

**Solution:**

\[ \Rightarrow \frac{x}{20} + \frac{y}{100} \geq 1, \quad \Rightarrow \frac{x}{40} + \frac{y}{60} \geq 1, \quad \Rightarrow \frac{x}{90} + \frac{y}{30} \geq 1 \]

\[ \text{at A (0, 100), } Z = 300 \]
\[ \text{at B (80, 300), } Z = 1080 \]
\[ \text{at C (180, 150), } Z = 810 \]
\[ \text{at D (90, 0), } Z = 180 \]

Optimal point

**Fig. 1.5**

**Example: 1.5**

Assume that the following specify a generalized linear programming problem:

\[ \text{Max } z = 300x_1 + 400x_2 \]

Subjected to

\[ 3x_1 + 6x_2 \leq 150 \]
\[ 5x_1 + 3x_2 \leq 180 \]
\[ 5x_1 + 4x_2 \geq 100 \]
\[ 2x_1 + 4x_2 \geq 80 \]

Solve it by using shortcut slope method.

**Solution:**

As it is maximization problem, do not consider equations having “\( \geq \)” sign.

\[ m = \frac{-3}{4} = -0.75; \quad m_1 = \frac{-3}{6} = -0.5; \quad m_2 = \frac{-5}{3} = -1.33 \]

\[ m \in (m_1, m_2) \quad \therefore \text{Optimum point is the intersection of line (1) and (2) i.e. (30, 10)} \]

\[ \Rightarrow \quad \text{max } Z = 13000 \]
Example 1.6

Assume that the following specify a generalized linear programming problem:

$$\text{Max } Z = 6x_1 + 4x_2$$

Subjected to

$$2x_1 + 3x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Solve it by using Graphical Method & slope methods.

Solution:

By graphical method:

$$\frac{x_1}{15} + \frac{x_2}{10} \leq 1$$
$$\frac{x_1}{8} + \frac{x_2}{12} \leq 1$$
$$\frac{x_1}{3} + \frac{x_2}{3} \geq 1$$
$$x_1, x_2 \geq 0$$

![Graphical Method Diagram](image)

- In these situations at the farthest point, objective function line superimposes over one of the constraint lines, so all the points falling on ‘DE’ line will be optimum.
- This is the case of alternate solution because the slope of objective function line is same as that of one of the constraints and objective function line is superimposing over the constraint line at the outermost point.

By slope method:

$$m = \frac{-6}{4} = -1.5$$

$$m_1 = \frac{-2}{3} = -0.66, \ m_2 = \frac{-3}{2} = -1.5$$

$$\therefore \quad m = m_2$$

It has alternative solutions.
Example: 1.7

Assume that the following specify a generalized linear programming problem:

\[ \text{Max } Z = 3x_1 + 2x_2 \]

Subject to

\[ x_1 - x_2 \leq 1 \]
\[ x_1 + x_2 \geq 3 \]
\[ x_1, x_2 \geq 0 \]

Solve it by using Graphical Method.

Solution:

\[ \frac{x_1}{3} + \frac{x_2}{3} \leq 1 \]
\[ \frac{x_1}{3} + \frac{x_2}{3} \geq 1 \]

Fig. 1.7

In problem feasible region extends to infinite, so for max \( Z \), it will be unbounded solution.

Example: 1.8

Assume that the following specify a generalized linear programming problem:

\[ \text{Max } Z = x_1 + x_2 \]

Subjected to

\[ x_1 + x_2 \leq 1 \]
\[ -3x_1 + x_2 \geq 3 \]
\[ x_1, x_2 \geq 0 \]

Solve it by using Graphical Method.

Solution:

\[ \frac{x_1}{1} + \frac{x_2}{1} \leq 1 \]
\[ \frac{x_1}{-1} + \frac{x_2}{3} \geq 1 \]

Fig 1.8

Regions (1) and (2) are not having any intersection region so there is no feasible region, so no solution.
1. Consider the following linear programming problem:
   \[ \text{Max. } Z = 2A + 3B, \text{ subject to } A + B \leq 10, 4A + 6B \leq 30, 2A + B \leq 17, \ A, B \geq 0 \]
   what can one say about the solution?
   (a) It may contain alternative optima (b) The solution will be unbounded
   (c) The solution will be degenerate (d) It cannot be solved by simplex method

2. In case of solution of a two variable linear programming problem by graphical method one constraint line comes parallel to the objective function line. Which one of the following is correct? The problem will have:
   (a) Infeasible solution (b) Unbounded solution
   (c) Degenerate solution (d) Infinite number of optimal solutions

3. Which of the following are correct in respect of graphically solved linear programming problems?
   1. The region of feasible solution has concavity property.
   2. The boundaries of the region are lines or planes.
   3. There are corners or extreme points on the boundary.
   Select the correct answer using the codes given below:
   (a) 1 and 3 (b) 2 and 3
   (c) 1 and 3 (d) 1, 2 and 3

4. If \( m \) is the number of constraints in a linear programming with two variables \( x \) and \( y \) and non-negativity constraints \( x > 0, y > 0 \); the feasible region in the graphical solution will be surrounded by how many lines?
   (a) \( m \) (b) \( m + 1 \)
   (c) \( m + 2 \) (d) \( m + 4 \)

5. Which one of the following is true in case of simplex method of linear programming?
   (a) The constants of constraints equation may be positive or negative.
   (b) Inequalities are not converted into equations.
   (c) It cannot be used for two-variable problems.
   (d) The simplex algorithm is an iterative procedure.

6. The linear programming is used for optimization problems which satisfy the following conditions:
   1. Objective function expressed as a linear function of variables.
   2. Resources are unlimited.
   3. The decision variables are interrelated and non-negative.
   Which of these statements is/are correct?
   (a) 1, 2 and 3 (b) 2 and 3 only
   (c) 1 only (d) 1 and 3 only

7. Consider the following statements:
   1. Resources limitations must be known.
   2. Relationship of variables must be known.
   Which of these statements must be satisfied to deal with the graphical techniques of linear programming effectively?
   (a) 1 only (b) 2 only
   (c) both 1 and 2 (d) neither 1 nor 2
8. The first algorithm for Linear Programming was given by:
   (a) Bellman  (b) Dantzig
   (c) Kulm  (d) Van Neumann

9. Let $y_1$ and $y_2$ be the decision variables of the dual and $v_1$ and $v_2$ be the slack variables of the dual of the given linear programming problem. The optimum dual variables are:
   (a) $y_1$ and $y_2$  (b) $y_1$ and $v_1$
   (c) $y_1$ and $v_2$  (d) $v_1$ and $v_2$

10. If at the optimum in a linear programming problem, a dual variable corresponding to particular primal constraint is zero, then it means that
   (a) Right hand side of the primal constraint can be altered without affecting the optimum solution.
   (b) Changing the right hand side of the primal constraint will disturb the optimum.
   (c) The objective function is unbounded.
   (d) The problem is degenerate.

11. Consider an objective function $z(x_1, x_2) = 3x_1 + 9x_2$ and constraints:
    
    $x_1 + x_2 \leq 8$
    $x_1 + 2x_2 \leq 4$
    $x_1 \geq 0$ and $x_2 \geq 0$

    The maximum value of the objective function is
    (a) 16  (b) 12
    (c) 18  (d) 72

12. A manufacturer produces two types of products, 1 and 2, at production levels of $x_1$ and $x_2$ respectively. The profit is given is $2x_1 + 5x_2$. The production constraints are:
    
    $x_1 + 3x_2 \leq 40$
    $3x_1 + x_2 \leq 24$
    $x_1 + x_2 \leq 10$
    $x_1 > 0$, $x_2 > 0$

    The maximum profit which can meet the constraints is
    (a) 29  (b) 38
    (c) 44  (d) 75

13. Consider the following Linear Programming Problem (LPP):
    Maximize: $z = 3x_1 + 2x_2$
    Subject to
    $x_1 \leq 4$
    $x_2 \leq 6$
    $3x_1 + 3x_2 \leq 18$
    $x_1 \geq 0$, $x_2 \geq 0$

    (a) The LPP has a unique optimal solution.
    (b) The LPP is infeasible.
    (c) The LPP is unbounded.
    (d) The LPP has multiple optimal solution.

14. Match List-I with List-II and select the correct answer using the codes given below the lists:

   **List-I**
   A. Linear programming
   B. Dynamic programming
   C. ‘C’ programming
   D. Integer programming

   **List-II**
   1. Ritchie
   2. Dantzig
   3. Bell
   4. Gomory
Codes:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>c</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>d</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

15. Linear programming model can be applied to:
   1. Line balancing problem
   2. Transportation problem
   3. Project management
Of these statements:
   (a) 1, 2 and 3 are correct       (b) 1 and 2 are correct
   (c) 2 and 3 are correct           (d) 1 and 3 are correct

16. The primal of a LP problem is maximization of objective function with 6 variables and 2 constraints. Which of the following correspond to the dual of the problem stated?
   1. It has 2 variables and 6 constraints.
   2. It has 6 variables and 2 constraints.
   3. Maximization of objective function.
   4. Minimization of objective function.
Select the correct answer using the codes given below:
   (a) 1 and 3       (b) 1 and 4
   (c) 2 and 3       (d) 2 and 4

17. Maximize: \( P = x + 12y \)
   Subjected to: 
   \[
   \begin{align*}
   20x + 10y & \leq 200 \\
   10x + 20y & \leq 120 \\
   10x + 30y & \leq 150 \\
   x \geq 0 \text{ and } y \geq 0
   \end{align*}
   \]
   (a) 50       (b) 66
   (c) 60       (d) 120

18. A feasible solution to the linear programming problem should
   (a) satisfy the problem constraints.
   (b) optimize the objective function.
   (c) satisfy the problem constraints and non-negativity restrictions.
   (d) satisfy the non-negativity restrictions.

19. Solution to \( z = 4x_1 + 6x_2 \)
    \[
    \begin{align*}
    x_1 + x_2 & \leq 4 \\
    3x_1 + x_2 & \leq 12 \\
    x_1, x_2 & \geq 0
    \end{align*}
    \]
    is
    (a) unique       (b) unbounded
    (c) degenerate   (d) infinite

20. Assumptions of linear programming include:
    1. Linearity
    2. Additivity
    3. Divisibility
    4. Certainty
Select the correct option
(a) 1 and 2  (b) 1 and 4  
(c) 1, 3 and 4  (d) 1, 2, 3 and 4

21. Divisibility assumption in linear programming employs
(a) resources can be divided among the product.
(b) product can be divided among the customers.
(c) decision variable may take an integer value.
(d) decision variable may take a fraction value.

22. Which of the following conditions are necessary of applying linear programming?
1. There must be a well-defined objective function.
2. The decision variables should be interrelated and non-negative.
3. The resources must be in limited supply.
(a) 1 and 2 only  (b) 1 and 3 only
(c) 2 and 3 only  (d) 1, 2 and 3

23. A variable which has no physical meaning, but is used to obtain an initial basic feasible solution to the linear programming problem is referred to as
(a) Basic variable  (b) Non-basic variable
(c) Artificial variable  (d) Basis

24. Consider the following statements regarding the characteristics of the standard form of a linear programming problem:
1. All the constraints are expressed in the form of equations.
2. The right-hand side of each constraint equation is non-negative.
3. All the decision variables are non-negative.
Which of these statements are correct?
(a) 1, 2 and 3  (b) 1 and 2
(c) 2 and 3  (d) 1 and 3

25. Which one of the following statements is NOT correct?
(a) Assignment model is a special case of a linear programming problem.
(b) In queuing models, Poisson arrivals and exponential services are assumed.
(c) In transportation problems, the non-square matrix is made square by adding a dummy row or a dummy column.
(d) In linear programming problems, dual of a dual is a primal.

26. Which of the following are correct in respect of graphically solved linear programming problems?
1. The region of feasible solution has concavity property.
2. The boundaries of the region are lines or planes.
3. There are corners or extreme points on the boundary.
Select the correct answer using the code given below:
(a) 1 and 2  (b) 2 and 3
(c) 1 and 3  (d) 1, 2 and 3

27. In case of solution of linear programming problem using graphical method, if the constraint line of one of the non-redundant constraints is parallel to the objective function line, then it indicates
(a) an infeasible solution  (b) a degenerate solution
(c) an unbound solution  (d) a multiple number of optimal solutions
28. In a linear programming problem, which one of the following is
(a) a point in the feasible region is not a solution to the problem is correct for graphical method.
(b) one of the corner points of the feasible region is not the optimum solution.
(c) any point in the positive quadrant does not satisfy the non-negativity constraint.
(d) the lines corresponding to different values of objective functions are parallel.

29. A company produces two types of paper towels, called regular and super soakers. Marketing has imposed a constraint that the total monthly production of regular should not be more than twice the production of super soakers. Letting \( x \) be the number of units of regular produced per month and \( y \) represents number of super soakers produced per month, appropriate constraints will be
(a) \( x \leq 2y \)  
(b) \( 2x \leq y \)  
(c) \( x + 0.5y \leq 0 \)  
(d) \( x - 0.5y \geq 0 \)

---

**ANSWERS**

**GRAPHICAL METHOD**

1. (a) 2. (d) 3. (b) 4. (c) 5. (d) 6. (d) 7. (c)
8. (b) 9. (d) 10. (c) 11. (c) 12. (a) 13. (a) 14. (c)
15. (b) 16. (b) 17. (d) 18. (c) 19. (a) 20. (d) 21. (c)
22. (d) 23. (c) 24. (a) 25. (c) 26. (b) 27. (d) 28. (a)

29. (b)
Simplex Method

2.1 INTRODUCTION

When there are more than two variables, graphical method cannot be used to solve optimization problems. As it was explained in the previous chapter that optimal solution exists always at the corner point at the feasible region, the simplex method is a systematic procedure at finding corner point solution and taking them for optimality. Simplex procedures are meant for profit maximization and if our objective is loss minimization then the problems have to be converted into profit maximization by multiplying the objective function by ‘−’ sign before starting the simplex procedures.

While solving problems using simplex methods, we take slack variables, surplus variables and artificial variables. After solving the simplex problem, we can do the verification test of the result to make sure that the answer is correct. In many practical problems we want to find not only an optimal solution but also want to determine what happens to this optimal solution when certain changes are made in the system. For this we do the sensitivity. While doing sensitivity analysis if negative value appears in solution matrix, the matrix is not an optimum one and we move towards optimality by dual simplex procedure.

While solving problems using simplex methods if many artificial variables are to be used we will write a dual problem for the given problem and then solve it by simplex procedure. The intermediate steps and fundamentals involved in simplex procedure are given in the following problems.

Slack variables are those which are added to the constraint equations to get equal to sign (‘=’)

Example 2.1

Assume that the following specify a generalized linear programming problem:

Maximize: \[ Z = x_1 - x_2 + 3x_3 \]

Subjected to

\[ 2x_1 - x_3 \leq 2 \]
\[ x_1 + x_2 - x_3 \leq 10 \]
\[ 2x_1 - 2x_2 + 3x_3 \leq 0 \]
\[ x_1, x_2, x_3 \geq 0 \]

Solve it by using Simplex Method.

Solution:

Let \( s_1, s_2, s_3 \) be slack variables then the constraints become

\[ 2x_1 - x_3 + s_1 = 2; \quad x_1 + x_2 - x_3 + s_2 = 10 \]
\[ 2x_1 - 2x_2 + 3x_3 + s_3 = 0; \quad x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \]

\[ \Rightarrow \quad \text{Max } Z = x_1 - x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 \]
Table 1:

- The coefficients of all the variables in the objective function are written in the first row against “$C_i$”.
- The coefficients of the slack variables in the objective function are written in the first column under “Basis”.
- The coefficients of all the variables in the constraints are written in the corresponding rows as shown below.
- The constants in the constraints are written under “b”
- The $E_i$ values of the variables are the sum of the products of the coefficients in the respective columns and the corresponding basis.
- e.g., $2x_0 + 1x_0 + 2x_0 = 0$

<table>
<thead>
<tr>
<th>Basis</th>
<th>$C_i$</th>
<th>$x_1$</th>
<th>$-1$</th>
<th>$x_2$</th>
<th>$3$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$b$</th>
<th>$\theta = \frac{b}{C_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0s_1$</td>
<td>2</td>
<td>0</td>
<td>$-1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0s_2$</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0s_3$</td>
<td>2</td>
<td>$-2$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_{ij}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j - C_i$</td>
<td>$-1$</td>
<td>1</td>
<td>$-3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- In $E_j - C_i$ row the value represents the corresponding profit on each machine.
- Select the maximum negative value in the pivot row, so that we select the max profit or min loss. The column corresponding to that value is called as entering column ($C_i$). This row is called pivot or key column.
- The $\theta$ values are calculated by dividing the values under $b$ with corresponding elements in the entering column.
- Select the minimum positive value (including ’0’) in the $\left(\frac{b}{C_i}\right)$ column,
- This represents idleness of the machine as shown in above table. The corresponding row is called the leaving row. This row is called pivot or key row.

Max $-ve$ value [means 3 units of profit] $\frac{d^2x}{dt^2} = -ve(max)$

- The element common in the entering column and leaving row is called pivotal value.
- The basis variable in the leaving row is replaced by the variable in entering column. i.e., $s_3$ leaves the basis, $x_3$ enters the basis ($s_3 \leftrightarrow x_3$).
Table 2:
- Pivotal value in row is made '1' by applying only row operations and all other values in the column are made zero and only by following row operations.
  \[
  R_{3\text{ new}} = R_{3\text{ old}} \div 3 \\
  R_{1\text{ new}} = R_{1\text{ old}} + R_{3\text{ new}} \\
  R_{2\text{ new}} = R_{2\text{ old}} + R_{3\text{ new}}
  \]
- The \(E_j\) and \(E_j - C_j\) values are calculated as explained earlier.

<table>
<thead>
<tr>
<th>Basis</th>
<th>(C_j)</th>
<th>1</th>
<th>(-1)</th>
<th>3</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>(b)</th>
<th>(\theta = \frac{b}{C_j})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 s_1)</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>(0 s_2)</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>-10</td>
<td></td>
</tr>
<tr>
<td>(3 x_3)</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(E_j)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(E_j - C_j)</td>
<td>-1</td>
<td>1</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

- The entering column is identified and the \(\theta\) values are calculated, then the leaving row are identified as explained earlier.

Note: The pivotal element should not be a negative number. So instead of “0” the next minimum positive number (here “30”) is taken for the leaving row.

Table 3:
- Pivotal value in row is made ‘1’ by applying only row operations and all other values in the column are made zero and only by following row operations.
\[ R_{2\text{ new}} = R_{2\text{ old}} + \frac{1}{3} \]
\[ R_{1\text{ new}} = R_{1\text{ old}} + \frac{2}{3} R_{2\text{ new}} \]
\[ R_{3\text{ new}} = R_{3\text{ old}} + \frac{2}{3} R_{2\text{ new}} \]

- The \( E_j \) and \( E_j - C_j \) values are calculated as explained earlier.

<table>
<thead>
<tr>
<th>Basic</th>
<th>( C_j )</th>
<th>( 1 )</th>
<th>( -1 )</th>
<th>( 3 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( b )</th>
<th>( 0 = b \times C_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>22</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_2 )</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_3 )</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_j )</td>
<td>7</td>
<td>-1</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_j - C_j )</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- As all the elements of \( E_j - C_j \) row are positive, it means it is an optimum matrix.
- The variable present in the basis are called basic variables.
- The variables which are not in the basis but are in the objective function are called non basic variables. (The non-basic variables are assigned zero solution.)
- So solution can be read out from matrix i.e.,

\[
\begin{align*}
  s_1 &= 22 \\
  x_2 &= 30 \\
  x_3 &= 20 \\
\end{align*}
\]

\[
\begin{align*}
  \text{Basic variables} & \quad s_2 = 0 \\
  \text{Non-basic variables} & \quad s_3 = 0 \\
  x_1 &= 0 \\
\end{align*}
\]

\[
\therefore \text{by putting } (x_1, x_2, x_3) = (0, 30, 20) \text{ in the objective function the maximum value of } Z = x_1 - x_2 + 3x_3 = 0 - 30 + 60 = 30
\]

**Example 2.2**

**Assume that the following specify a generalized linear programming problem:**

**Maximize:** \( Z = 45x + 40y \)

**Subjected to**

\[
\begin{align*}
  2x + y &\leq 90 \\
  x + 2y &\leq 80 \\
  x + y &\leq 50 \\
  x, y &\geq 0 \\
\end{align*}
\]

**Solve it by using Simplex Method.**

**Solution:**

Let \( s_1, s_2, s_3 \) be slack variables then the constraints become

\[
\begin{align*}
  2x + y + s_1 &= 90 \\
  x + 2y + s_2 &= 80 \\
  x + y + s_3 &= 50 \\
  x, y, s_1, s_2, s_3 &\geq 0 \\
\end{align*}
\]

\( \Rightarrow \)

\[
Z = 45x + 40y + 0s_1 + 0s_2 + 0s_3
\]

- The above information is tabulated.
- \( E_j \) and \( E_j - C_j \) are calculated.
- The entering column is identified and the \( 0 \) values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
Table-1

$s_1$ leaves the basis and $x$ enters into the basis.
- The following row operations are performed:
  \[ R_{1_{\text{new}}} = R_{1_{\text{old}}}/2 \]
  \[ R_{2_{\text{new}}} = R_{2_{\text{old}}} - R_{1_{\text{new}}} \]
  \[ R_{3_{\text{new}}} = R_{3_{\text{old}}} - R_{1_{\text{new}}} \]
- Then the table 1 becomes table 2.
- $E_j$ and $E_j - C_j$ are calculated.
- The entering column is identified and the $\theta$ values are calculated, then the leaving row is identified.
- The pivotal element/value is identified from the table.
- $s_3$ leaves the basis, and $y$ enters the basis.

Table-2

- The following row operations are performed:
  \[ R_{1_{\text{new}}} = R_{1_{\text{old}}}/2 \]
  \[ R_{2_{\text{new}}} = R_{2_{\text{old}}} - R_{3_{\text{new}}} \]
  \[ R_{3_{\text{new}}} = 2R_{3_{\text{old}}} \]
- Then the table 2 becomes

Table-3
• $E_j$ and $E_j - C_j$ are calculated.
• ‘$E_j - C_j$’ Values present in the matrix are positive. So it is an optimum matrix.
• So solution can be read out from matrix i.e.,

\[
\begin{align*}
  x &= 40 \\
  y &= 10 \\
  s_1 &= 0 \\
  s_2 &= 20
\end{align*}
\]

Basic variables $s_1 = 0$ Non-basic variables $s_2 = 0$

.: by putting $(x, y) = (40, 10)$ in the objective function the maximum value of $Z = 45x + 40y = 2200$

• **Surplus variables** are those which are subtracted from the constraint equation to get equal to sign (‘=’)

• **Artificial variables** are those which are added to the constraint equations whenever we introduce a surplus variable in order to get the initial feasible solution, which incurs a heavy loss in the above function. And so we assign a large penalty “−M” to these variables in the objective function.

**Example: 2.3**

Assume that the following specify a generalized linear programming problem:

Minimize:

\[ Z = 4x_1 + x_2 \]

Subjected to

\[
\begin{align*}
  3x_1 + 4x_2 &\geq 20 \\
  x_1 + 5x_2 &\leq 15 \\
  x_1, x_2 &\geq 0
\end{align*}
\]

Solve it by using Big M Method or artificial variable Method.

**Solution:**

By introducing surplus variable $s_1$, artificial variable $a_1$ and slack variable $s_2$ the problem may be written as follows:

Maximize:

\[ Z = -4x_1 - x_2 + 0s_1 - Ma_1 + 0s_2 \]

Subject to:

\[
\begin{align*}
  3x_1 + 4x_2 - s_1 + a_1 &= 20 \\
  x_1 + 5x_2 + s_2 &= 15 \\
  x_1, x_2, s_1, s_2, a_1 &\geq 0
\end{align*}
\]

• The above information is tabulated.

• In any constraint if artificial variable is present that should enter first in the basis. (Once the artificial variable leaves the basis it will never enter again, so from the next simplex table onwards that variable can be deleted permanently.)

• $E_j$ and $E_j - C_j$ are calculated.

• The entering column is identified and the $\theta$ values are calculated, then the leaving row are identified.

• The pivotal element/value is identified from the table.

<table>
<thead>
<tr>
<th>Basis</th>
<th>$C_j$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$0$</th>
<th>$-M$</th>
<th>$a_1$</th>
<th>$b$</th>
<th>$\theta = \frac{b}{C_j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-Ma_1$</td>
<td>3</td>
<td>4</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>20</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 $s_2$</td>
<td>1</td>
<td>5 $^*$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>15</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j$</td>
<td>$-3M$</td>
<td>$-4M$</td>
<td>$M$</td>
<td>0</td>
<td>$-M$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j - C_j$</td>
<td>$-3M + 4$</td>
<td>$-4M + 1$</td>
<td>$M$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table-1
• $S_2$ leaves the basis and $X_2$ enters the basis.
• The following row operations are performed:
  \[ R_{2_{\text{new}}} = \frac{R_{2_{\text{old}}}}{5}; \quad R_{1_{\text{new}}} = R_{1_{\text{old}}} - 4 \left( R_{2_{\text{new}}} \right) \]
• Then the table 1 becomes

<table>
<thead>
<tr>
<th>Basis</th>
<th>$C_j$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>0</th>
<th>0</th>
<th>$-M$</th>
<th>$b$</th>
<th>$\theta = \frac{b}{C_j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-Mx_1$</td>
<td>$1/5$</td>
<td>0</td>
<td>$-1$</td>
<td>$-4/5$</td>
<td>1</td>
<td>8</td>
<td>$40/11$</td>
<td></td>
</tr>
<tr>
<td>$-x_2$</td>
<td>1</td>
<td>0</td>
<td>1/5</td>
<td>0</td>
<td>3</td>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j$</td>
<td>$-11 \frac{M-1}{5}$</td>
<td>$-1$</td>
<td>$M$</td>
<td>$\frac{4M-1}{5}$</td>
<td>$M$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j - C_j$</td>
<td>$-11 \frac{M+19}{5}$</td>
<td>0</td>
<td>$M$</td>
<td>$\frac{4M-1}{5}$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-2**

• $E_j$ and $E_j - C_j$ are calculated.
• The entering column is identified and the $\theta$ values are calculated, then the leaving row is identified.
• The pivotal element/value is identified from the table.
• $a_1$ leaves the basis and $x_1$ enters the basis.
• As $a_1$ leaves the basis, it will never come back to the basis again, so that particular column is removed from matrix.
• The following row operations are performed:
  \[ R_{1_{\text{new}}} = 5 \left( R_{1_{\text{old}}} \right); \quad R_{2_{\text{new}}} = R_{2_{\text{old}}} - \left( R_{1_{\text{new}}} \right) \]
• Then the table 2 becomes table 3.
• $E_j$ and $E_j - C_j$ are calculated.
• It is an optimum matrix since all the values obtained in $E_j - C_j$ row are positive

<table>
<thead>
<tr>
<th>Basis</th>
<th>$C_j$</th>
<th>$-4$</th>
<th>$-1$</th>
<th>0</th>
<th>0</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4x_1$</td>
<td>1</td>
<td>0</td>
<td>$-5/11$</td>
<td>$-4/11$</td>
<td>$40/11$</td>
<td></td>
</tr>
<tr>
<td>$-x_2$</td>
<td>0</td>
<td>1</td>
<td>$1/11$</td>
<td>$3/11$</td>
<td>$25/11$</td>
<td></td>
</tr>
<tr>
<td>$E_j$</td>
<td>$-4$</td>
<td>$-1$</td>
<td>$19/11$</td>
<td>$13/11$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j - C_j$</td>
<td>0</td>
<td>0</td>
<td>$19/11$</td>
<td>$13/11$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-3**

• So solution can be read it out from matrix i.e.,
  \[
  x_1 = \frac{40}{11} \quad s_1 = 0 \\
  x_2 = \frac{25}{11} \quad s_2 = 0 \\
  a_1 = 0
  \]
• by putting $(x_1, x_2) = \left( \frac{40}{11}, \frac{25}{11} \right)$ in the objective function the maximum value of
  \[
  Z = -4x_1 - x_2 = \frac{185}{11}
  \]
Example: 2.4

Assume that the following specify a generalized linear programming problem:

Minimum: \[ Z = 3x_1 + 2x_2 \]

Subject to:
\[ 2x_1 + x_2 \leq 2 \]
\[ 3x_1 + 4x_2 \geq 12 \]
\[ x_1, x_2 \geq 0, \]

Solve it by using Big M Method.

Solution:

By introducing slack variable \( s_1 \), surplus variable \( s_2 \), and artificial variable \( a_2 \), the problem may be written as follows:

Maximize:
\[ Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 - Ma_2 \]

Subject to:
\[ 2x_1 + x_2 + s_1 = 2 \]
\[ 3x_1 + 4x_2 - s_2 + a_2 = 12 \]
\[ x_1, x_2, s_1, s_2, a_2 \geq 0 \]

- The above information is tabulated.
- In any constraint if artificial variable is present that should enter first in the basis. (Once the artificial variable leaves the basis it will never enter again, so from the next simplex table onwards that variable can be deleted permanently.)

<table>
<thead>
<tr>
<th>Basis</th>
<th>(-3 )</th>
<th>(-2 )</th>
<th>(0)</th>
<th>(0)</th>
<th>(-M)</th>
<th>(a_2)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(-Ma_2)</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>-1</td>
<td>12</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>(E_j)</td>
<td>-3</td>
<td>-4</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-M</td>
<td></td>
</tr>
<tr>
<td>(E_j - C_j)</td>
<td>(-3M + 3)</td>
<td>(-4M + 2)</td>
<td>0</td>
<td>(M)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table-1

- \(E_j\) and \(E_j - C_j\) are calculated.
- The entering column is identified and the \(\theta\) values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- \(s_1\) leaves the basis, \(x_2\) enters the basis.
- The following row operations are performed
\[ R_2 \text{ new } \rightarrow R_2 \text{ old } - 4R_1 \text{ new } \]
- Then the table 1 becomes

<table>
<thead>
<tr>
<th>Basis</th>
<th>(-3 )</th>
<th>(-2 )</th>
<th>(0)</th>
<th>(0)</th>
<th>(-M)</th>
<th>(a_2)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x_2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(-Ma_2)</td>
<td>-5</td>
<td>0</td>
<td>-4</td>
<td>-1</td>
<td>1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>(E_j)</td>
<td>5M - 4</td>
<td>-2</td>
<td>4M - 2</td>
<td>(M)</td>
<td>-M</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(E_j - C_j)</td>
<td>5M - 1</td>
<td>0</td>
<td>4M - 2</td>
<td>(M)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table-2

- It is an optimum matrix since all the values obtained in \(E_j - C_j\) row are positive.
- But the optimum matrix contains artificial variable, this indicates infeasible solution i.e., (there is no feasible region).
If we do the same problem by graphical method we get
\[
\frac{x_1}{1} + \frac{x_2}{2} \leq 1 \\
\frac{x_1}{4} + \frac{x_2}{3} \geq 1
\]

**Note:** After selecting the entering column, if all the values appear to be negative, it indicates that the solution is unbounded.

**Example: 2.5**

**Assume that the following specify a generalized linear programming problem:**

**Maximize:**  
\[ Z = 6x_1 + 4x_2 \]

**Subject to**  
\[
2x_1 + 3x_2 \leq 30 \\
3x_1 + 2x_2 \leq 24 \\
x_1 + x_2 \geq 3 \\
x_1, x_2 \geq 0
\]

**Solve it by using Big \( M \) Method.**

**Solution:**

By introducing slack variables \( s_1, s_2 \) surplus variable \( s_3 \) and artificial variable \( a_i \) the problem may be written as follows:

**Maximize:**  
\[ Z = 6x_1 + 4x_2 + 0.s_1 + 0.s_2 + 0.s_3 - Ma_1 \]

**Subject to**  
\[
2x_1 + 3x_2 + s_1 = 30 \\
3x_1 + 2x_2 + s_2 = 24 \\
x_1 + x_2 - s_3 + a_1 = 3 \\
x_1, x_2, s_1, s_2, s_3, a_1 \geq 0
\]

- The above information is tabulated.
- In any constraint if artificial variable is present that should enter first in the basis.
- \( E_j \) and \( E_j - C_j \) are calculated.
- The entering column is identified and the 0 values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table

<table>
<thead>
<tr>
<th>Basis</th>
<th>( C_j )</th>
<th>( 6 )</th>
<th>( 4 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( -M )</th>
<th>( b )</th>
<th>( \theta = \frac{b}{C_j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 s_1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>15</td>
</tr>
<tr>
<td>0 s_2</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>24</td>
<td>8</td>
</tr>
<tr>
<td>M a_1</td>
<td>[1]</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>E_j</td>
<td>-M</td>
<td>-M</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>-M</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E_j - C_j</td>
<td>-M</td>
<td>-M</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-1**
- $s_1$ leaves the basis, $x_2$ enters the basis.
- The following row operations are performed:
  
  \[
  R_{1_{\text{new}}} = R_{1_{\text{old}}} - 2R_{3_{\text{new}}} \\
  R_{2_{\text{new}}} = R_{2_{\text{old}}} - 3R_{3_{\text{new}}}
  \]

- Then the table 1 becomes.
- $E_j$ and $E_j - C_j$ are calculated.
- The entering column is identified and the $\theta$ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- $s_2$ leaves the basis, $s_3$ enters the basis.

<table>
<thead>
<tr>
<th>Basis</th>
<th>6</th>
<th>4</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>b</th>
<th>$\theta = \frac{b}{C_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>24</td>
<td>12</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>$E_j$</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j - C_j$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-2**

- The following row operations are performed:
  
  \[
  R_{2_{\text{new}}} = (R_{2_{old}}) \frac{1}{3} \\
  R_{3_{new}} = R_{3_{old}} - 2R_{2_{new}} \\
  R_{3_{new}} = R_{3_{old}} + R_{2_{new}}
  \]

- Then the table 2 becomes.

<table>
<thead>
<tr>
<th>Basis</th>
<th>6</th>
<th>4</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>b</th>
<th>$\theta = \frac{b}{C_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>5/3</td>
<td>1</td>
<td>-2/3</td>
<td>0</td>
<td>14</td>
<td>42/5</td>
<td></td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>1/3</td>
<td>1</td>
<td>5</td>
<td>-15</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>2/3</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>8</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$E_j$</td>
<td>6</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j - C_j$</td>
<td>0</td>
<td>(0)</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-3**

- $E_j$ and $E_j - C_j$ are calculated.
- It is an optimum matrix since all the values obtained in ‘$E_j - C_j$’ row are positive.
- So solution can be read out from matrix i.e.,

\[
\begin{align*}
    s_1 &= 14 \quad x_2 = 0 \\
    s_2 &= 5 \quad s_3 = 0 \quad \Rightarrow \quad X_1 = \\
    x_1 &= 8 \quad a_1 = 0
\end{align*}
\]

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    s_1 \\
    s_2 \\
    s_3
\end{pmatrix} = \begin{pmatrix}
    8 \\
    0 \\
    14 \\
    0 \\
    5
\end{pmatrix}
\]
Normally in simplex, zero will appear in $E_j - C_j$ row corresponding to every basic variable. If it so happens that zero appears below a non basic variables it means there are alternate solutions.

Here $x_2$ is a non basic variable and is having ‘0’ in row. So, this is the case of alternate solution.

To get the alternative solution the entering column is identified as $x_2$ is having 0’ in and is not in the basis and the θ values are calculated, then the leaving row are identified.

The pivotal element/value is identified from the table.

The following row operations are performed:

$$
R_{1_{\text{new}}} = 3(R_{1_{\text{old}}})/5 \\
R_{2_{\text{new}}} = R_{2_{\text{old}}} - (R_{1_{\text{new}}})/3 \\
R_{3_{\text{new}}} = R_{3_{\text{old}}} + (R_{1_{\text{new}}})/3
$$

Then the table 3 becomes.

<table>
<thead>
<tr>
<th>Basis</th>
<th>$C_j$</th>
<th>6</th>
<th>4</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td></td>
<td>1</td>
<td>1/5</td>
<td>3/5</td>
<td>1</td>
<td>0</td>
<td>127/5</td>
</tr>
<tr>
<td>$s_3$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>39/5</td>
</tr>
<tr>
<td>$x_1$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>2/5</td>
<td>3/5</td>
<td>0</td>
<td>12/5</td>
</tr>
<tr>
<td>$E_j$</td>
<td></td>
<td>6</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$E_j - C_j$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table-4

$E_j$ and $E_j - C_j$ are calculated.

It is an alternative optimum matrix since all the values obtained in $E_j - C_j$ row are positive.

So solution can be read it out from matrix i.e.,

$$
X_2 = \begin{pmatrix}
 x_1 \\
 x_2 \\
 s_1 \\
 s_2 \\
 s_3 \\
\end{pmatrix} = \begin{pmatrix}
 12/5 \\
 45/2 \\
 0 \\
 0 \\
 39/5 \\
\end{pmatrix}
$$

Here alternate solution means infinitely many solutions may be there because the objective function line is coinciding with the constraint line at the farthest point. The infinite number of solutions is represented as

$$
x = (\lambda)x_2 + (1-\lambda)x_2
$$

$y \in (0, 1)$

Example: 2.6

Assume that the following specify a generalized linear programming problem:

Maximum: $Z = 2x_1 + x_2$

Subject to: $4x_1 + 3x_2 \leq 12$

$4x_1 + x_2 \leq 8$

$4x_1 - x_2 \leq 8$

$x_1, x_2 \geq 0$

Solve it by using Simplex Method.
Solution:

By introducing slack variables \( s_1, s_2 \) and \( s_3 \) the problem may be written as follows:

\[
\text{Maximum:} \quad Z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 \\
\text{Subjected to:} \quad 
\begin{align*}
4x_1 + 3x_2 + s_1 &= 12 \\
4x_1 + x_2 + s_2 &= 8, \\
4x_1 - x_2 + s_3 &= 8. \\
x_1, x_2, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

The above information is tabulated.

<table>
<thead>
<tr>
<th>Basis</th>
<th>( C_j )</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>b</th>
<th>( \theta = \frac{b}{C_j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( s_1 )</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0 ( s_2 )</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0 ( s_3 )</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( E_j )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( E_j - C_j )</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

**Table-1**

- \( E_j \) and \( E_j - C_j \) are calculated.
- The entering column is identified and the \( \theta \) values are calculated.
- In the ‘\( \theta \)’ column when two minimum positive values are same and we are not able to decide which row to enter it is called degeneracy in simplex.

- To find degenerate solution of simplex we have to find minimum \( \left( \frac{s_1}{C_j} \right) \).
- Here in \( s_1 \) column both \( \left( \frac{s_1}{C_j} \right) \) are same. But in \( s_2 \) column Min \( \left( \frac{1}{4}, \frac{0}{4} \right) \) is ‘0’. So \( s_3 \) is identified as leaving row.
- The pivotal element/value is identified from the table.
- \( s_3 \) leaves the basis, \( x_1 \) enters the basis.
- The following row operations are performed:
  \[
  \begin{align*}
  R_{3\text{new}} &= (R_{3\text{old}})/4 \\
  R_{1\text{new}} &= R_{1\text{old}} - 4(R_{3\text{new}}) \\
  R_{2\text{new}} &= R_{2\text{old}} - 4(R_{3\text{new}})
  \end{align*}
  \]
- Then the table 1 becomes.

<table>
<thead>
<tr>
<th>Basis</th>
<th>( C_j )</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>b</th>
<th>( \theta = \frac{b}{C_j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( s_1 )</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0 ( s_2 )</td>
<td>0</td>
<td>2*</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( 2 \times x_1 )</td>
<td>0</td>
<td>1</td>
<td>-1/4</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>2</td>
<td>-8</td>
</tr>
<tr>
<td>( E_j )</td>
<td>2</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>4</td>
<td>-16</td>
<td></td>
</tr>
<tr>
<td>( E_j - C_j )</td>
<td>0</td>
<td>-3/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table-2**
• \( E_i \) and \( E_i - C_j \) are calculated.
• The entering column is identified and the \( \theta \) values are calculated, then the leaving row are identified.
• The pivotal element/value is identified from the table.
• \( s_2 \) leaves the basis, \( x_2 \) enters the basis.
• The following row operations are performed:
  \[
  \begin{align*}
  R_{2,\text{new}} &= \frac{R_{2,\text{old}}}{2} \\
  R_{1,\text{new}} &= R_{1,\text{old}} - 4(R_{2,\text{new}}) \\
  R_{3,\text{new}} &= R_{3,\text{old}} + \frac{R_{2,\text{new}}}{4}
  \end{align*}
  \]
• Then the table 1 becomes:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( C_j )</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>( b )</th>
<th>( \theta = \frac{b}{C_j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( s_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 ( x_2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 ( x_1 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/8</td>
<td>1/8</td>
<td>2</td>
<td>16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_i )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>3/4</td>
<td>-1/4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_i - C_j )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3/4</td>
<td>-1/4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-3**

• \( E_i \) and \( E_i - C_j \) are calculated.
• The entering column is identified and the \( \theta \) values are calculated, then the leaving row are identified.
• The pivotal element/value is identified from the table.
• \( s_1 \) leaves the basis, \( s_3 \) enters the basis.
• The following row operations are performed:
  \[
  \begin{align*}
  R_{2,\text{new}} &= \frac{R_{2,\text{old}} + (R_{1,\text{new}})}{2} \\
  R_{3,\text{new}} &= R_{3,\text{old}} - \frac{(R_{1,\text{new}})}{8}
  \end{align*}
  \]
• Then the table 3 becomes:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( C_j )</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( s_2 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>1 ( s_2 )</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2 ( x_1 )</td>
<td>1</td>
<td>0</td>
<td>-1/8</td>
<td>3/8</td>
<td>0</td>
<td>3/2</td>
<td></td>
</tr>
<tr>
<td>( E_i )</td>
<td>2</td>
<td>1</td>
<td>1/4</td>
<td>1/4</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_i - C_j )</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>1/4</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-4**

• \( E_i \) and \( E_i - C_j \) are calculated.
• It is an optimum matrix since all the values obtained in ‘\( E_i - C_j \)’ row are positive.
So solution can be read it out from matrix i.e.,

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    s_1 \\
    s_2 \\
    s_3
\end{bmatrix}
= \begin{bmatrix}
    3 \\
    2 \\
    0 \\
    0 \\
    4
\end{bmatrix}
\]

By putting \((x_1, x_2) = \left(\frac{3}{2}, 2\right)\) in the objective function the maximum value of \(Z = 2x_1 + x_2 = 5\)

2.2 VERIFICATION OF SIMPLEX RESULTS

If the product of the matrix under the slack or surplus variables in the final optimal table and the matrix of constants in the given constraints is equal to the solution matrix then the solution is correct.

Example 2.7

Assume that the following specify a generalized linear programming problem:

Maximum: \(Z = 3x_1 + 2x_2\)

Subject to:

\[\begin{align*}
    x_1 + x_2 &\leq 6 \\
    2x_1 + x_2 &\leq 8 \\
    -x_1 + x_2 &\leq 1, x_2 &\leq 2
\end{align*}\]

Solve it by using Simplex Method and verify the result.

Solution:

By introducing slack variable \(s_1, s_2, s_3\) and \(s_4\) the problem may be written as follows:

Maximum: \(Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4\)

\[\begin{align*}
    x_1 + x_2 + s_1 &= 6 \\
    2x_1 + x_2 + s_2 &= 8 \\
    -x_1 + x_2 + s_3 &= 1 \\
    x_2 + s_4 &= 2 \\
    x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{align*}\]

The above information is tabulated.

<table>
<thead>
<tr>
<th>Basis</th>
<th>(C_i)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
<th>(s_4)</th>
<th>(b)</th>
<th>(\theta = \frac{b}{C_i})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s_1)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>(s_2)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>4</td>
<td>-</td>
</tr>
<tr>
<td>(s_3)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>(s_4)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(E_j)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(E_j - C_i)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table-1
• \( E_j \) and \( E_j - C_j \) are calculated.
• The entering column is identified and the \( \theta \) values are calculated, then the leaving row are identified.
• The pivotal element/value is identified from the table.
• \( s_3 \) leaves the basis, \( x_4 \) enters the basis.
• The following row operations are performed:
  \[
  R_{2_{\text{new}}} = \frac{(R_{3_{\text{old}}})}{2}, \\
  R_{1_{\text{new}}} = R_{1_{\text{old}}} - R_{2_{\text{new}}} \times R_{3_{\text{new}}}, \\
  R_{3_{\text{new}}} = R_{3_{\text{old}}} + R_{2_{\text{new}}}
  \]
• Then the table 1 becomes.
• \( E_j \) and \( E_j - C_j \) are calculated.
• The entering column is identified and the \( \theta \) values are calculated, then the leaving row are identified.
• The pivotal element/value is identified from the table.
• \( s_4 \) leaves the basis, \( x_3 \) enters the basis.

<table>
<thead>
<tr>
<th>Basis</th>
<th>( C_j )</th>
<th>3</th>
<th>( x_1 )</th>
<th>2</th>
<th>( x_2 )</th>
<th>0</th>
<th>( s_1 )</th>
<th>0</th>
<th>( s_2 )</th>
<th>0</th>
<th>( s_3 )</th>
<th>0</th>
<th>( s_4 )</th>
<th>( b )</th>
<th>( \theta = \frac{b}{C_j} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( s_1 )</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 3 )</td>
<td>( x_1 )</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0 )</td>
<td>( s_3 )</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>10/3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0 )</td>
<td>( s_4 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_j )</td>
<td>3</td>
<td>3/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_j - C_j )</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-2**

• The following row operations are performed:
  \[
  R_{1_{\text{new}}} = R_{1_{\text{old}}} - \frac{(R_{4_{\text{new}}})}{2}, \\
  R_{2_{\text{new}}} = R_{2_{\text{old}}} - \frac{(R_{4_{\text{new}}})}{2}, \\
  R_{3_{\text{new}}} = R_{3_{\text{old}}} - \frac{3(R_{4_{\text{new}}})}{2}
  \]
• Then the table 2 becomes table 3.
• \( E_j \) and \( E_j - C_j \) are calculated.

<table>
<thead>
<tr>
<th>Basis</th>
<th>( C_j )</th>
<th>3</th>
<th>( x_1 )</th>
<th>2</th>
<th>( x_2 )</th>
<th>0</th>
<th>( s_1 )</th>
<th>0</th>
<th>( s_2 )</th>
<th>0</th>
<th>( s_3 )</th>
<th>0</th>
<th>( s_4 )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( s_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
<td>-1/2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 3 )</td>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
<td>-1/2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 0 )</td>
<td>( s_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>-3/2</td>
<td>1</td>
<td>-3/2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2 )</td>
<td>( x_2 )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_j )</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_j - C_j )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table-3**
• It is an optimum matrix since all the values obtained in ‘$E_j - C_j$’ row are positive.

From table,
\[
\begin{bmatrix}
3x = \\
\begin{bmatrix}
s_1 \\
x_1 \\
s_3 \\
x_2
\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}
1 & -1/2 & 0 & -1/2 \\
0 & 1/2 & 0 & -1/2 \\
0 & 1/2 & 1 & -3/2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{bmatrix}
\]

From table,
\[
\begin{bmatrix}
1 & B = \\
\begin{bmatrix}
6 \\
8 \\
1 \\
2
\end{bmatrix}
\end{bmatrix}
\]

• Here as $A \times B = X$ the solution is correct.

### 2.3 SENSITIVITY ANALYSIS OF SIMPLEX

The process of determining the effects of these changes without solving the problem from the very beginning is known as sensitivity analysis.

**Example: 2.8**

In (Q.No. 2.7) determine the effect in solution if the constant ‘6’ in first constraint is changed to ‘7’.

**Solution:**

The simplex problem can be expressed in the form of matrices as follows:

\[
Z = \begin{bmatrix}
3 & 2
\end{bmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

\[
x_1 + 2x_2 \leq 6
\]
\[
2x_1 + x_2 \leq 8
\]
\[
-x_1 + x_2 \leq 1 \Rightarrow
\]
\[
x_2 \leq 2
\]

Its optimum simplex matrix is

<table>
<thead>
<tr>
<th>Basis</th>
<th>$C_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 $s_1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-1/2$</td>
<td>0</td>
<td>$-1/2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3 $x_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$1/2$</td>
<td>0</td>
<td>$-1/2$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0 $s_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$1/2$</td>
<td>1</td>
<td>$-3/2$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2 $x_2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$E_j$</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>$3/2$</td>
<td>0</td>
<td>$1/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_j - C_j$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$3/2$</td>
<td>0</td>
<td>$1/2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
A = \begin{bmatrix}
1 & -1/2 & 0 & -1/2 \\
0 & 1/2 & 0 & -1/2 \\
0 & 1/2 & 1 & -3/2 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad X = \begin{bmatrix}
\begin{cases}
x_2 = 1 \\
x_1 = 3 \\
s_3 = 2 \\
s_4 = 2
\end{cases}
\end{bmatrix}
\]
If we change constant value in problem the matrices will be become

\[
\begin{align*}
    x_1 + 2x_2 &\leq 7 \\
    2x_1 + x_2 &\leq 8 \\
    -x_1 + x_2 &\leq 1 \\
    x_2 &\leq 2
\end{align*}
\]

\[
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix} \leq
\begin{pmatrix}
    7 \\
    8 \\
    1 \\
    2
\end{pmatrix}
\]

Note: By changing the constants as negative values appeared in \( A \times B \) matrix, the matrix is not an optimum one and sowe move towards optimality by dual simplex procedure.

\[
A \times B =
\begin{pmatrix}
    1 & -1/2 & 0 & -1/2 \\
    0 & 1/2 & 0 & -1/2 \\
    0 & 1/2 & 1 & -3/2 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    7 \\
    8 \\
    1 \\
    2
\end{pmatrix}
=
\begin{pmatrix}
    2 \\
    3 \\
    2 \\
    2
\end{pmatrix}
\]

\[
\therefore
x_2 = 2, x_1 = 3, s_3 = 2, s_4 = 2
\]

Example: 2.9

In (Q.No. 2.7) determine the effect in solution if the constant ‘6 and 8’ in constraints are changed to ‘5 and 10’.

Solution:

The simplex problem can be expressed in the form of matrices as follows:

\[
Z = \begin{pmatrix}
    3 \\
    2
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

\[
\begin{align*}
    x_1 + 2x_2 &\leq 6 \\
    2x_1 + x_2 &\leq 8 \\
    -x_1 + x_2 &\leq 1 \\
    x_2 &\leq 2
\end{align*}
\]

\[
\begin{pmatrix}
    1 & 2 \\
    2 & 1
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix} \leq
\begin{pmatrix}
    6 \\
    8 \\
    1 \\
    2
\end{pmatrix}
\]

Its optimum simplex matrix is:

<table>
<thead>
<tr>
<th>Basis</th>
<th>( C_j )</th>
<th>( 3 )</th>
<th>( 2 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( 0 )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( x_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>0</td>
<td>-1/2</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>-1/2</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>( x_3 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
<td>-3/2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( E_j - C_j )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>1/2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If we change constant value in problem the matrices will be become

\[
\begin{align*}
    x_1 + 2x_2 &\leq 5 \\
    2x_1 + x_2 &\leq 10 \\
    -x_1 + x_2 &\leq 1 \\
    x_2 &\leq 2
\end{align*}
\]

\[
\begin{pmatrix}
    1 & 2 \\
    2 & 1
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix} \leq
\begin{pmatrix}
    5 \\
    10 \\
    1 \\
    2
\end{pmatrix}
\]

Note: By any change in the constant, if all the values of \( A \times B \) appears to be “+ve” then, matrix is still an optimum one but the solution will change as below.
2.4 DUAL SIMPLEX PROCEDURE

- The above matrices is tabulated as table 1.
- We will not calculate ‘0’ column instead we will select max negative value from ‘b’ column as leaving row.

<table>
<thead>
<tr>
<th>Basis</th>
<th>( C_j )</th>
<th>3</th>
<th>2</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( s_i )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>0</td>
<td>-1/2</td>
<td>-1</td>
</tr>
<tr>
<td>3 ( x_1 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>-1/2</td>
<td>4</td>
</tr>
<tr>
<td>0 ( s_2 )</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>1/2</td>
<td>1</td>
<td>-3/2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2 ( x_2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( E_j )</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_j - C_j )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>1/2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table-2.1
- Select the min positive value excluding zero in \( E_j - C_j \) row as entering column.
- Get the pivotal element (whichmay be negative also).
- \( s_3 \) leaves the basis, \( s_1 \) enters the basis
- The following row operations are performed:

\[
R_{1\text{ new}} = R_{1\text{ old}} - 2(R_{3\text{ new}})/3,
\]
\[
R_{2\text{ new}} = R_{2\text{ old}} + (R_{3\text{ new}})/3,
\]
\[
R_{4\text{ new}} = R_{4\text{ new}} + 2(R_{3\text{ new}})/3
\]
- Then the table 1 becomes

<table>
<thead>
<tr>
<th>Basis</th>
<th>( C_j )</th>
<th>3</th>
<th>2</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( s_i )</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3 ( x_1 )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>0 ( s_2 )</td>
<td>0</td>
<td>0</td>
<td>-3</td>
<td>2</td>
<td>-1/2</td>
<td>0</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>2 ( x_2 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( E_j )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_j - C_j )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table-2.2
- Thus the optimum solution will change as: \( x_1 = 5, x_2 = 0, x_3 = 6, s_4 = 2. \)
Example: 2.10

Convert the following simplex problem into a dual problem

Minimum: \[ Z = 5x_1 + 2x_2 + x_3 \]

Subjected to:
- \[ 2x_1 + 3x_2 + x_3 \geq 20 \]
- \[ 6x_1 + 8x_2 + 5x_3 \geq 30 \]
- \[ 7x_1 + x_2 + 3x_3 \geq 40 \]
- \[ x_1 + 2x_2 + 4x_3 \geq 50 \]
- \[ x_1, x_2, x_3 \geq 0 \]

Solution:

The simplex problem can be expressed in the form of matrices as follows:

\[
\begin{align*}
\text{Minimum,} & \quad Z = A(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) \cdot \begin{bmatrix} 2 & 3 & 1 \\ 6 & 8 & 5 \\ 7 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 20 \\ 30 \\ 40 \\ 50 \end{bmatrix} \\
\text{Note: If primal problem is} & \quad \min Z = AX \\
\text{then Dual Problem is} & \quad \max Z = C^T y \\
\text{So the dual for the given problem becomes:} & \\
\text{Maximum,} & \quad Z = (\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}) \cdot \begin{bmatrix} 2 & 6 & 7 & 1 \\ 3 & 8 & 1 & 2 \\ 1 & 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}
\end{align*}
\]

Note-1:
1. For every simplex there exists another problem which gives exactly same solution called dual.
2. Dual of dual will again be primal problem.
3. Before starting the normal simplex makes sure that constants of the constraints are positive. When the negative sign appears multiply the constraint by -ve sign.

Note-2:
1. Suppose there are “n” number of equations, if number of variables are more than number of equations (Rank of the matrix is less than “n”), then there will be infinite number of solutions.
2. If number of variables are equal to the number of equations (Rank is equal to “n”), then it will be a unique solution.
3. If numbers of equations are more than variables, then there will be no solution.