A Handbook on Instrumentation Engineering

Contains well illustrated formulae & key theory concepts for GATE, PSUs & OTHER COMPETITIVE EXAMS

MADE EASY Publications
A Handbook on Instrumentation Engineering

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Director’s Message

During the current age of international competition in Science and Technology, the Indian participation through skilled technical professionals have been challenging to the world. Constant efforts and desire to achieve top positions are still required.

I feel every candidate has ability to succeed but competitive environment and quality guidance is required to achieve high level goals. At MADE EASY, we help you to discover your hidden talent and success quotient to achieve your ultimate goals. In my opinion GATE & PSU’s exams are tool to enter in to main stream of Nation serving. The real application of knowledge and talent starts, after you enter in to the working system. Here in MADE EASY you are also trained to become winner in your life and achieve job satisfaction.

MADE EASY alumni have shared their winning stories of success and expressed their gratitude towards quality guidance of MADE EASY. Our students have not only secured All India First Ranks in ESE, GATE and PSU entrance examinations but also secured top positions in their career profiles. Now, I invite you to become alumni of MADE EASY to explore and achieve ultimate goal of your life. I promise to provide you quality guidance with competitive environment which is far advanced and ahead than the reach of other institutions. You will get the guidance, support and inspiration that you need to reach the peak of your career.

I have true desire to serve Society and Nation by way of making easy path of the education for the people of India.

After a long experience of teaching in Instrumentation Engineering over the period of time MADE EASY team realised that there is a need of good Handbook which can provide the crux of Instrumentation Engineering in a concise form to the student to brush up the formulae and important concepts required for GATE, PSUs and other competitive examinations. This handbook contains all the formulae and important theoretical aspects of Instrumentation Engineering. It provides much needed revision aid and study guidance before examinations.

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A Handbook on

Instrumentation Engineering

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Engineering Mathematics

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MATRIX

Principal Diagonal: In a square matrix all elements $a_{ij}$ for which $i = j$ are elements of principal diagonal.

Matrices

1. **Upper Triangular matrix**: A square matrix in which all the elements below the principle diagonal are zero.
2. **Lower Triangular Matrix**: A square matrix in which all the elements above the principle diagonal are zero.
3. **Diagonal Matrix**: A square matrix in which all the elements other than the elements of principle diagonal are zero.
4. **Scalar Matrix**: A diagonal matrix with all elements of principle diagonal being same.
5. **Idempotent Matrix**: ‘$A$’ is square matrix i.e. $A^2 = A$.
6. **Involuntary Matrix**: ‘$A$’ is square matrix i.e.$A^2 = I$.
7. **Nilpotent Matrix**: ‘$A$’ is square matrix i.e. $A^m = 0$ where $m$ is the least positive integer and $m$ is also called as Index of class of Nilpotent matrix $A$.
8. **Transpose Matrix**: $A^T$ is transpose matrix of matrix $A$. $A^T$ can be obtained by switching the rows as columns and columns as rows of $A$.
9. **Symmetric Matrix**: ‘$A$’ is a square matrix i.e. $A^T = A$.
10. **Skew-Symmetric Matrix**: ‘$A$’ is a square matrix i.e. $A^T = -A$.
11. **Orthogonal Matrix**: ‘$A$’ is a orthogonal matrix i.e. $A^T = A^{-1}$ or $AA^T = I = A^TA$.
12. **Conjugate Matrix of $A$ ($\bar{A}$) or ($\sim A$)**: ‘$A$’ is any matrix, by replacing the elements by corresponding conjugate complex numbers the matrix obtained is conjugate of ‘$A$’.

Example:

$$A = \begin{bmatrix} 2+3i & 4+7i & 5 \\ 2i & 3 & 9-i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2-3i & 4-7i & 5 \\ -2i & 3 & 9+i \end{bmatrix}$$
13. Transpose Conjugate Matrix \((A^\theta)\) or \((A^*)\): \((\bar{A})^T\).

14. Hermitian Matrix: ‘\(A\)’ is a square matrix i.e. \(A^\theta = A\).
   All diagonal elements of hermitian matrix are real number and all off-diagonal elements above and below the principle diagonal must be conjugate of each other i.e. \(a_{ij} = \overline{a_{ji}}\).

   **Example:**
   \[
   \begin{bmatrix}
   2 & 3 - 4i \\
   3 + 4i & 5
   \end{bmatrix}
   \]

15. Skew-Hermitian Matrix: ‘\(A\)’ is a square matrix i.e. \(A^\theta = -A\).
   All diagonal elements of Skew-Hermitian matrix are purely imaginary or zero and all off-diagonal elements above and below the principle diagonal must be conjugate of each other with opposite sign. i.e. \(a_{ij} = -\overline{a_{ji}}\).

   **Example:**
   \[
   \begin{bmatrix}
   2i & 3 - 4i \\
   -3 - 4i & 5
   \end{bmatrix}
   \]

16. Unitary Matrix: ‘\(A\)’ is a square matrix i.e. \(A^\theta = A^{-1}\) or \(AA^\theta = I = A^\theta A\).

17. Boolean Matrix: Any matrix with only elements ‘0’ or ‘1’

18. Sparse Matrix: A matrix ‘\(A\)’ in which more number of elements are zeros.

19. Singular and Non-singular Matrix: A square matrix ‘\(A\)’ is singular if \(|A| = 0\), and non singular if \(|A| \neq 0\). Only non-singular matrices have inverse.

20. Adjoint Matrix: Transpose of cofactors matrix. i.e. \(\text{Adj}(A) = (\text{Cof}(A))^T\)

**Properties of Matrices**

- \(A + B = B + A\) (Commutative)
- \((A + B) + C = A + (B + C)\) (Associative)
- \(AB \neq BA\) (Not commutative)
- \((AB)C = A(BC)\) (Associative)
- \(A(B + C) = AB + AC\) (Distributive)
- \(A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)\)
- \(A(\text{Adj} A) = (\text{Adj} A)A = |A|I_n\)
- \(\text{Adj} (AB) = (\text{Adj} B) \cdot (\text{Adj} A)\)
- \(A^{-1} = \frac{\text{Adj} A}{|A|}; |A| \neq 0\)
- \((A^{-1})^{-1} = A\) and \((A^{-1})^T = (A^T)^{-1}\)
- \((AB)^{-1} = B^{-1}A^{-1}\).
- If \(A\) is a square matrix of order \(n\) then \(\text{det } A = (\text{det } A)^{n-1} = |A|^{n-1}\)
  and \(|\text{Adj } (\text{Adj } A)| = |A|^{(n-1)^2}\)
- If \(|A| \neq 0\) then \(|A^{-1}| = \frac{1}{|A|}\)
- If \(A_{n \times n}\) matrix then \(|KA| = |K|A|
- If \(A\) is square matrix then
  1. \(A + A^T\) is always symmetric
  2. \(A - A^T\) always skew-symmetric
- If \(A\) and \(B\) are symmetric then
  1. \(A + B\) is also symmetric.
  2. \(A - B\) is also symmetric.
  3. \(AB + BA\) is symmetric
  4. \(AB - BA\) is skew-symmetric
  5. \(A^n\) and \(B^n\) are symmetric
- If \(A\) and \(B\) are skew-symmetric then,
  1. \(A + B\) is also skew-symmetric.
  2. \(A - B\) is also skew-symmetric.
  3. \(A^n\) and \(B^n\) are symmetric, if ‘\(n\)’ is even
  4. \(A^n\) and \(B^n\) are skew-symmetric, if ‘\(n\)’ is odd
- The determinant of orthogonal matrix and unitary matrix \(A\) has absolute value ‘1’.
- If \(A_{m \times n}\) and \(B_{n \times p}\) then product of \(AB\) requires
  1. \(mnp\) multiplications
  2. \(m(n - 1)p\) additions
  3. for each entry, \(n\) multiplications and \((n - 1)\) additions.
- \((A^T)^T = A\), \((kA)^T = k(A^T)\)
- \((A + B)^T = A^T + B^T\), \((AB)^T = B^TA^T\)
- **Rank of Matrix (r(A))**: It is the order of its largest non-vanishing (non-zero) minor of the matrix.
- Rank is equal to the number of linearly independent rows or columns in the matrix.
The system of linear equation $AX = B$ has a solution (consistent) iff rank of $A = \text{Rank of } (A\mid B)$.

The system $AX = B$ has

(i) A unique solution iff $\text{Rank } (A) = \text{Rank } (A\mid B) = \text{Number of variables}$.

(ii) Infinitely many solutions $\iff \text{Rank } (A) = \text{Rank } (A\mid B) < \text{number of variables}$.

(iii) No solution if $\text{Rank } (A) \neq \text{Rank } (A\mid B)$ i.e. $\text{Rank } (A) < \text{Rank } (A\mid B)$.

The system $AX = 0$ has

(i) Unique solution (zero solution or trivial solution) if $\text{Rank } (A) = \text{number of variables}$.

(ii) Infinitely many number of solutions (non-trivial solutions) if $\text{Rank } (A) < \text{number of variables}$.

If $\text{Rank } (A) = r$, and number of variables = $n$ then, the number of linearly independent infinite solutions of $AX = 0$ is $(n - r)$.

In the system of homogenous linear equation $AX = 0$

(i) If $A$ is singular then the system possesses non-trivial solution (i.e. infinite solution).

(ii) If $A$ is non-singular then the system possesses trivial (zero) solution (i.e. unique solution).

Rank of a diagonal matrix = Number of non-zero elements in diagonal.

If $A$ and $B$ are two matrices

(i) $r(A + B) \leq r(A) + r(B)$

(ii) $r(A - B) \geq r(A) - r(B)$

(iii) $r(AB) \leq \text{min } \{r(A), r(B)\}$

If a matrix $A$ has rank ‘$R$’, then $A$ contains ‘$R$’ linearly independent vectors (row/column).

The system of homogeneous linear equations such that number of unknowns (or variables) exceeds the number of equations necessarily possesses a non-zero solution.

**Eigen Value**

Let ‘$A$’ be a square matrix of order $n$ and $\lambda$ be a scalar then $|A - \lambda I| = 0$ is the characteristic equation of $A$. The roots of characteristic equation are called eigen values/lantent roots/Characteristic roots.

- The set of eigen values of matrix is called “spectrum of matrix”.
- A matrix of order $n$ will have $n$ latent roots not necessarily distinct.
**Eigen Vector**

Corresponding to each eigen value \( \lambda \), there exists a non-zero solution \( X \) such that \((A - \lambda I)X = 0\) then \( X \) is eigen vector/latent/vector/characteristic vector of \( A \).

**Properties of Eigen Values**

- Sum of eigen values of a matrix = sum of elements of principal diagonal (trace).
  \[ \Sigma \lambda_i = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = \text{Trace of } A \]
- Product of eigen values = Determinant of matrix.
  \[ \Pi \lambda_i = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n = \det(A) \]
- If \( \lambda \) is eigen value of \( A \) then \( \frac{1}{\lambda} \) is eigen value of \( A^{-1} \). (provided \( \lambda \neq 0 \) i.e. \( A \) is non-singular).
- Eigen values of \( A \) and \( A^T \) are same.
- If \( \lambda \) is eigen value of orthogonal matrix then \( \frac{1}{\lambda} \) is also its eigen value \[ (: \quad A^T = A^{-1}) \]
- If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigen values of matrix \( A \), then
  
  (i) \( \lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m \) are eigen values of matrix \( A^m \).
  
  (ii) \( \lambda_1 + K, \lambda_2 + K, \ldots, \lambda_n + K \) are eigen values of \( A + K I \)
  
  (iii) \( (\lambda_1 - K)^2, (\lambda_2 - K)^2, \ldots, (\lambda_n - K)^2 \) are eigen values of \( (A - K I)^2 \)
  
  (iv) \( K\lambda_1, K\lambda_2, \ldots, K\lambda_n \) are eigen values of \( KA \).
- The eigen values of symmetric matrix are real.
- The eigen values of skew-symmetric matrix are either purely imaginary or zero.
- The modulus of the eigen values of orthogonal and unitary matrices = 1.
- If a matrix is either lower or upper triangular or diagonal then the principal diagonal elements themselves are the eigen values.
- Zero is eigen value of a matrix iff the matrix is singular.
Differential Equation

Order of differential equation: It is the order of the highest derivative appearing in it.

Degree of Differential Equation: It is the degree of the highest derivative occurring in it, after expressing the equation free from radicals and fractions as far as derivatives are concerned.

Differential Equations of First Order First Degree

Equations of first order and first degree can be expressed in the form $y' = f(x, y)$. Following are the different ways of solving equations of first order and first degree:

1. Variable Separable: $f(x)dx + g(y)dy = 0$
   $$\int f(x)dx + \int g(y)dy = c$$ is the solution.

2. Homogenous Equation: $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$
   - To solve a homogeneous equation, substitute $y = Vx$
     $$\frac{dy}{dx} = V + x \frac{dV}{dx}$$
   - Separate the variable $V$ and $x$ and integrate.

Equations Reducible to Homogenous Equation

The differential equation:
$$\frac{dy}{dx} = \frac{ax + by + x'}{a'x + b'y + c'}$$

This is non-homogeneous but can be converted to homogeneous equation.

Case I: If $\frac{a}{a'} \neq \frac{b}{b'}$

Substitute $x = X + h$, $y = Y + k(h$ and $K$ are constants)
Solve for $h$ and $k$, we get
$$ah + bk + c = 0$$
$$a'h + b'k + c' = 0$$
$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$
Case II : If \( \frac{a}{a'} = \frac{b}{b'} \)

\[
\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m} \text{ (say)}
\]

\[
\frac{dy}{dx} = \frac{(ax + by) + C}{m(ax + by) + C'}
\]

Substitute \( ax + by = t \), so that

\[
\frac{dt}{dx} = \frac{b(t + C)}{mt + C'} + a
\]

Solve by variable separable method

3. Linear Equations : The standard form of a linear equation of first order

\[
\frac{dy}{dx} + P(x)y = Q(x), \text{ where } P \text{ and } Q \text{ are functions of } x.
\]

Second order linear equation \( \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \)

It is commonly known as “Leibnitz’s linear equations”

Integrating factor, \( \text{I.F.} = e^{\int Pdx} \)

\[
ye^{\int Pdx} = \int Q \cdot (\text{I.F.}) + dx + C \Rightarrow y(\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + C
\]

Note: The degree of every linear differential equation is always one but if the degree of the differential equation is one, then it need not be linear.

For Ex. : \( \frac{d^4y}{dx^4} + 3x^2\left(\frac{dy}{dx}\right)^3 + y^{2003} = 0 \)

Bernoulli’s Equation

\[
\frac{dy}{dx} + Py = Qy^n \text{ where } P \& Q \text{ are functions of } x \text{ only}
\]

Divide by \( y^n \)

\[
y^{-n}\frac{dy}{dx} + Py^{1-n} = Q
\]

Substituting, \( y^{1-n} = z \), we get

\[
\frac{dz}{dx} + (1-n)Pz = Q(1-n)
\]

This is a linear equation and can be solved easily.
4. Exact Differential Equations: The equations are given as
\[ M(x, y)dx + N(x, y)dy = 0 \]
The necessary and sufficient condition for the differential equations
\[ Mdx + Ndy = 0 \]
to be exact is \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \).

Solution of exact differential equation
\[ \int_y M \, dx + \int_{y \text{ is constant}} \text{(terms of } N \text{ not containing } x) \, dy = C \]

Equation Reducible to Exact Equation

**Integrating Factor**: Sometimes an equation which is not exact may become so on multiplication by some function known as Integrating factgor (I.F.).

**Case I**: Finding by inspection

1. \( x \, dy + y \, dx = d(x, y) \)
2. \( \frac{x dy - y dx}{xy} = d \left[ \log \left( \frac{y}{x} \right) \right] \)
3. \( \frac{x dy - y dx}{x^2 + y^2} = d \left[ \tan^{-1} \left( \frac{y}{x} \right) \right] \)
4. \( \frac{xdy - y dx}{x^2} = d \left( \frac{y}{x} \right) \)
5. \( \frac{xdy - y dx}{y^2} = -d \left( \frac{y}{x} \right) \)
6. \( \frac{xdy - y dx}{x^2 - y^2} = d \left[ \frac{1}{2} \log \left( \frac{x + y}{x - y} \right) \right] \)

**Case II**: When \( Mdx + Ndy = 0 \) is homogenous in \( x \) and \( y \) and \( Mx + Ny \neq 0 \), then \( I.F. = \frac{1}{Mx + Ny} \).

**Case III**: If the equation \( f_1(x, y) y dx + f_2(x, y) y dy = 0 \) and \( Mx - Ny \neq 0 \), then \( I.F. = \frac{1}{Mx - Ny} \).

**Case IV**: If the \( Mdx + Ndy = 0 \) and \( \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = f(x) \), then
\[ I.F. = e^{\int f(x) \, dx} \]

**Case V**: If the equation \( Mdx + Ndy = 0 \) and \( \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = f(y) \), then
\[ I.F. = e^{\int f(y) \, dy} \].
Linear Differential Equation with Constant Coefficients

\[ \frac{d^n y}{dx^n} + \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_n y = X \]

The equation can be written as \((D^n + k_1 D^{n-1} + \ldots + k_n) \ y = X\)
(where \(D = d/dx\))
\(f(D)y + X; \ f(D) = 0\) is called auxiliary equation.

Rules for Finding Complimentary Function

**Case I:** If all the roots of AE are real and different \((D - m_1) (D - m_2) \ldots \ (D - m_n) y = 0\)
So, the solution is \(y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}\)

**Case II:** If two roots are equal, i.e. \(m_1 = m_2\), then \(y = (C_1 + C_2 x) e^{m_1 x}\)
Similarly, if \(m_1 = m_2 = m_3\), \(y = (C_1 + C_2 x + C_3 x^2) e^{m_1 x}\)

**Case III:** If one pair of roots are imaginary, i.e. \(m_1 = \alpha + i\beta\), \(m_2 = \alpha - i\beta\), then \(y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)\)

**Case IV:** If two pairs of root are imaginary, i.e. repeated imaginary root, then \(y = e^{\alpha x}[(C_1 x + C_2) \cos \beta x + (C_3 x + C_4) \sin \beta x]\)

Rules for Finding Particular Integral

\[ P.I. = \frac{1}{D^n + k_1 D^{n-1} + \ldots + k_n} \frac{X}{f(D)} = \frac{1}{f(D)} X \]

**Case I:**
When \(X = e^{a x}\)
\[ P.I. = \frac{1}{f(D)} e^{a x} \quad \text{put} \quad D = a \quad [f(a) \neq 0] \]
\[ P.I. = x \frac{1}{f'(D)} e^{a x} \quad \text{put} \quad D = a \quad [f'(a) \neq 0, f(a) = 0] \]
\[ P.I. = x^2 \frac{1}{f''(D)} e^{a x} \quad \text{put} \quad D = a \quad [f'(a) = 0, f'(a) = 0, f''(a) \neq 0] \]

**Case II:**
When \(X = \sin(ax + b)\) or \(\cos(ax + b)\)
\[ P.I. = \frac{1}{\phi(D^2)} \sin(ax + b) \quad \text{put} \quad D^2 = -a^2 [\phi(-a^2) \neq 0] \]
\[ = x \frac{1}{\phi'(D^2)} \sin(ax + b) \quad \text{put } D^2 = -a^2[\phi'(-a^2) \neq 0, \phi(-a^2) = 0] \]
\[ = x \frac{1}{\phi''(D^2)} \sin(ax + b) \quad \text{put } D^2 = -a^2[\phi''(-a^2) \neq 0, \phi'(-a^2) = 0, \phi(-a^2) = 0] \]

**Case III**:  
When \( X = x^m, m \) being positive integer

\[ P.I. = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m \]
\[ = f(D) \left[ 1 + \frac{1}{f(D)} \right]^{-1} x^m = f(D)[1 - f(D) + f^2(D) - f^3(D) + \ldots.] x^m \]

**Case IV**:  
When \( X = e^{ax} \) \( V \) where \( V \) is function of \( x \)

\[ P.I. = \frac{1}{f(D)} e^{ax} V \]
\[ = e^{ax} \frac{1}{f(D + a)} V \text{ then evaluate } \frac{1}{f(D + a)} V \text{ as in Case I, II, III.} \]

**Case V**:  
When \( X = xV(x) \)

\[ P.I. = \frac{1}{f(D)} xV(x) = \left[ x - \frac{f'(D)}{f(D)} \right] \frac{1}{f(D)} V(x) \]

**Case VI**:  
When \( X \) is any other function of \( x \)

\[ P.I. = \frac{1}{f(D)} X \]

Factorize \( f(D) = (D - m_1)(D - m_2) \ldots (D - m_n) \) and resolve \( 1/f(D) \) into partial fractions and then apply, \[ \frac{1}{D - a} X = e^{ax} \int X e^{-ax} \, dx \] on each terms

Complete Solution: \( y = \text{C.F.} + \text{P.I.} \)

**Cauchy-Euler Equation (Homogenous Linear Equation)**

\[ x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_{n-1} \frac{dy}{dx} + k_n y = X \]
Substitute \( x = e^t \)

\[
x \frac{dy}{dx} = Dy
\]

\[
x^2 \frac{d^2y}{dx^2} = D(D - 1)y
\]

\[
x^3 \frac{d^3y}{dx^3} = D(D - 1)(D - 2)y
\]

After substituting these differentials, the Cauchy-Euler equation results in a linear equation with constant coefficients.

**Legendre’s Linear Equation**

\[
(ax + b)^n \frac{d^n y}{dx^n} + k_1(ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_n y = X
\]

\[
ax + b = e^t
\]

\[
\Rightarrow \quad t = \ln(ax + b)
\]

\[
(ax + b) \frac{dy}{dx} = aDy
\]

\[
(ax + b)^2 \frac{d^2y}{dx^2} = a^2 D(D - 1)y
\]

\[
(ax + b)^3 \frac{d^3y}{dx^3} = a^3 D(D - 1)(D - 2)y
\]

After substituting these differentials, the Legendre’s equation results in a linear equation with constant coefficients.

**Partial Differential Equation**

\[ z = f(x, y) \]

\[
\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t
\]

**Homogenous Linear Equation with Constant Coefficients**

\[
\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots + k_n \frac{\partial^n z}{\partial y^n} = f(x, y) \quad \text{→ this is called homogenous because all terms containing derivative is of same order.}
\]

\[
D^n + k_1 D^{n-1} + \ldots + k_n D^n = f(x, y) \quad \left( \text{where } D = \frac{\partial}{\partial x} \text{ and } D' = \frac{\partial}{\partial y} \right)
\]

\[ f(D, D') = f(x, y) \]
Step 1: Find the C.F.
1. Write A.E.
   
   \[ m^n + k_1m^{n-1} + \ldots + k_n = 0 \text{ where } m = \frac{D}{D'} \text{. Roots are } m_1, m_2, \ldots, m_n. \]

   \[ m^n + k_0m^{n-1} + k_n = 0 \text{, where } m = \frac{D}{D'} \text{. Roots are } m_1, m_2, \ldots, m_n. \]

2. CF = \( f_1(y + m_1x) + f_2(y + m_2x) + \ldots \quad m_1, m_2 \text{ are distinct} \)
   
   CF = \( f_1(y + m_1x) + xf_2(y + m_2x) + f_3(y + m_3x) \ldots m_1, m_2, m_3 \text{ two equal roots} \)
   
   CF = \( f_1(y + m_1x) + xf_2(y + m_2x) + x^2f_3(y + m_3x) \ldots m_1, m_2, m_3 \text{ three equal roots} \)

Step 2: Finding P.I.

\[ P.I. = \frac{1}{f(D, D')} f(x, y) \]

1. When \( F(ax + by) = e^{ax + by}, \) put \([D = a, D' = b]\)

2. When \( F(x, y) = \sin(mx + ny), \) put \([D^2 = -m^2, DD' = -mn, D'^2 = -n^2]\)

3. When \( F(x, y) = x^m y^n, \) P.I. = \( \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n. \)

4. When \( F(x, y) \) is any function of \( x \) and \( y, \) \( \frac{1}{f(D, D')} f(x, y), \) resolve

\( \frac{1}{f(D, D')} \) into partial fractions considering \( f(D, D') \) as a function of \( D \) alone and operate each partial fraction of \( f(x, y) \) remembering that

\[ \frac{1}{(D - mD')} f(x, y) = \int f(x, c - mx) dx \text{ where } c \text{ is replaced by } y + mx \text{ after integration}. \]

\[ \text{-----} \]
### MEAN, MEDIAN AND MODE

- **Mean** ($\bar{X}$)
  \[ \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum f_i} \]

- **Median**
  \[ \frac{x_{n/2} + \left( \frac{x_{n/2}}{2} \right)}{2} \text{; } n \text{ is even} \]
  \[ = x_{n+1/2} \text{; } n \text{ is odd} \]
  \[ = L_{\text{med}} + \left( \frac{N/2 - F}{f} \right) \times w \]

where $L_{\text{med}}$ = lower limit of median class
$N$ = $\Sigma f_i$
$F$ = Cumulative frequency upto the median class
   (cumulative frequency of the preceding class)
$f$ = Frequency of the median class

**Mode**
Value of 'x' corresponding to maximum frequency.
\[ L_{\text{mode}} + \frac{f_m - f_1}{(2f_m - f_1 - f_2)} \times h \]

where $L_{\text{mode}}$ = lower limit of modal class
$f_m$ = frequency of modal class
$f_1$ = preceding frequency of modal class
$f_2$ = Following frequency of modal class

**Note:**
- Mode = 3 Median – 2 Mean [for Asymmetric distribution]
- Mean = Mode = Median [for Symmetric distribution]
AXIOMS OF PROBABILITY

Let A and B be two events. Then

1. \( P(\overline{A}) = 1 - P(A) \)
2. \( P(\emptyset) = 0; \emptyset \) is the empty set
3. \( P(A \cap B) = P(A) - P(A \cap B) \)
4. \( P(A \cap B) = P(A \cup \overline{B}) \)
5. \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)
6. \( P(A \cup B) = P(A) + P(B); \) mutually exclusive events.
7. \( P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B) \)
8. \( P(A \cap B) = P(A) \cdot P(B); \) independent events.
9. \( P(A \cap B) = \emptyset; \) mutually exclusive events.
10. \( P(S) = 1; \) \( S \) is sample space.
11. \( \max(0, P(A) + P(B) - 1) \leq P(A \cap B) \leq \min(P(A), P(B)) \)
12. \( \max(P(A), P(B)) \leq P(A \cup B) \leq \min(1, P(A) + P(B)) \)
13. \( \max(0, P(A_1) + P(A_2) + \ldots + P(A_n) - (n - 1)) \leq P(A_1 \cap A_2 \cap \ldots \cap A_n) \)
    \( \leq \min(P(A_1), P(A_2), \ldots, P(A_n)) \)
14. \( \max(P(A_1), P(A_2), \ldots, P(A_n)) \leq P(A_1 \cup A_2 \cup \ldots \cup A_n) \)
    \( \leq \min(1, P(A_1) + P(A_2) + \ldots + P(A_n)) \)

15. \( P(A|B) = \frac{P(A \cap B)}{P(B)} \)

16. \( P(A|B) = P(A); \) independent events.
17. \( P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1) \cdot P(E_2) \cdot \ldots \cdot P(E_n); \) independent events.

18. Rule of total probability: \( P(X) = \sum_{i=1}^{n} P(E_i) \cdot P(X|E_i) \)

19. Baye’s Theorem:

\[
P(E_1|X) = \frac{P(E_1) \cdot P(X|E_1)}{P(E_1) \cdot P(X|E_1) + P(E_2) \cdot P(X|E_2) + P(E_3) \cdot P(X|E_3)}
\]

In general,

\[
P(E_i|X) = \frac{P(E_i) \cdot P(X|E_i)}{\sum_{j=1}^{n} P(E_j) \cdot P(X|E_j)}
\]
RANDOM VARIABLE (STOCHASTIC VARIABLE)

Random variable assigns a real number to each possible outcome.

Let $X$ be a discrete random variable, then
1. $F(x) = P(X \leq x)$ is called distribution function $\sum_{i=0}^{n} P(i)$ of $X$.

2. Mean or Expectation of $X = \mu = E(X) = \sum_{i=1}^{n} x_i P(x_i)$

3. Variance of $X = \sigma^2 = E(X^2) - [E(X)]^2 = \sum_{i=1}^{n} (x_i - \mu)^2 P(x_i)$

4. Standard deviation of $X = \sigma = \sqrt{\text{Variance}}$

5. $\sum_{i=1}^{n} P(x_i) = 1$

Types of Random Variables

1. **Discrete Random Variable**: “Finite set of values” or “Countably infinite”.

2. **Continuous Random Variable (Non-discrete)**: “Infinite number of uncountable values”.

Discrete Distributions

1. **Binomial Distribution**: The probability that the event will happen exactly $r$ times in $n$ trials i.e. $r$ successes and $n-r$ failures will occur.

   $P(X = r) = \binom{n}{r} p^r q^{n-r}$

   Mean = $E(x) = np$

   Variance $(\sigma^2) = V(x) = npq = np(1 - p)$

   $S.D \ (\sigma) = \sqrt{npq} = \sqrt{np(1-p)}$

   Where $r = 0, 1, ..., n$, $q = 1 - p$, $n$ = fixed number of trials,

   $p =$ probability of success

2. **Poisson Distribution**:

   $$P(X = x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^x}{x!}; \quad x = 0, 1, 2, ..., \infty$$

   Where $X =$ Discrete random variable

   $\lambda =$ Parameter of distribution (positive constant)

   • Mean $(\mu) =$ Variance $(\sigma^2) = \lambda$

   • $S.D = \sqrt{\lambda}$

   Poisson distribution is a limiting case of binomial distribution as $n \rightarrow \infty$ and $p \rightarrow 0.$
CONTINUOUS DISTRIBUTION

Let \( X \) be a continuous random variable. Then

(i) **Density functions:**

\[
\begin{align*}
\mathcal{P}(X \leq a) &= \int_{-\infty}^{a} f(x) \, dx \\
\mathcal{P}(a \leq X \leq b) &= \int_{a}^{b} f(x) \, dx
\end{align*}
\]

(ii) **Mean** \( E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx \)

(iii) **Variance of** \( X = \text{V}(X) = \int_{-\infty}^{\infty} [x - E(x)]^2 \cdot f(x) \, dx = E(x^2) - (E(x))^2 \)

\[
\begin{align*}
&= \int_{-\infty}^{\infty} x^2 f(x) \, dx - \left( \int_{-\infty}^{\infty} x f(x) \, dx \right)^2
\end{align*}
\]

(iv) \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

1. **Uniform Distribution (Rectangular Distribution)**

   (i) **Density function:**

   \[
   f(x) = \frac{1}{b - a}; \quad a \leq x \leq b
   \]

   \[= 0; \text{ otherwise} \]

   (ii) **Cumulative function:**

   \[
   \mathcal{P}(X \leq x) = \int_{-\infty}^{x} f(x) \, dx = \begin{cases} 
   0 & \text{if } x < a \\
   \frac{x - a}{b - a} & \text{if } a \leq x \leq b \\
   1 & \text{if } x > b
   \end{cases}
   \]

   (iii) **Mean** \( (\mu) = (a + b)/2 = E(X) \)

   (iv) **Variance** \( (\sigma^2) = (b - a)^2/12 \)
2. Normal Distribution

(i) Density function \( f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} ; -\infty \leq x \leq \infty, \sigma > 0, -\infty < \mu < \infty \)

(ii) Normal distribution is symmetrical

(iii) Mean = \( \mu \); Variance = \( \sigma^2 \)

(iv) \( f(x) \geq 0 \) for all \( x \)

(v) \( \int_{-\infty}^{\infty} f(x) \cdot dx = 1 \)

(vi) \( P(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} ; -\infty \leq Z \leq \infty \)

and \( Z = \frac{x - \mu}{\sigma} \)

\( Z = \frac{x - np}{\sqrt{npq}} \) (when approximating binomial by normal)

\( Z = \) Standard normal variate

3. Exponential Distribution

(i) Density function:

\[
\begin{align*}
    f(x) &= \lambda \cdot e^{-\lambda x} \quad ; \quad x > 0 \\
    &= 0 \quad ; \quad \text{Otherwise}
\end{align*}
\]

(i) Mean (\( \mu \)) = \( \frac{1}{\lambda} = SD(\sigma) \)

(i) Variance (\( \sigma^2 \)) = \( \frac{1}{\lambda^2} \)